1-1-1971

# Evaluation of buildup factors by the transmission matrix method

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by

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A Thesis Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
MASTER OF SCIENCE

Major Subject: Nuclear Engineering



Iowa State University
Ames, Iowa

1971

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# I. INTRODUCTION

A reactor core emits radiations, mainly gamma ray and neutron flux, in all directions. A portion of these radiations and their secondary emissions would reach the outside of the reactor vessel and become hazardous to the surrounding and reactor personnel. In order to minimize the radiation effects from undue exposure, it is necessary to enclose the reactor core with suitable shieldings.

In most thermal reactors, the basic neutron flux distribution is generally confined in the core and reflector. The bulk of shielding problem is that of gamma attenuation.

Photon interactions with matter can be classified into many types of reactions. In shielding studies, only three types of gamma interactions with matter need to be considered. They are the photoelectric effect, Compton scattering and electron-positron pair production. Gamma attenuation involving only photoelectric and pair production effect can be considered as absorption processes and thus can be described accurately by an exponentially decreasing relation. However, in energy region around 1 Mev, Compton scattering is the dominant process among the three. Compton scattering gives rise to secondary photons; hence a correction factor must be applied to the exponentially decreasing relation of gamma attenuation. This correction factor, known as the buildup factor, is therefore an important parameter for shielding studies.

Buildup factors have been calculated by the moments method for infinite homogeneous media. However, since reactor shields are finite, heterogeneous media, the moments method is not directly applicable. Buildup factors have also been calculated by Monte Carlo methods. By their very nature of random sampling, the computer time required by Monte Carlo methods is prohibitively high for parametric studies. Furthermore, it is not well suited for deep penetration studies. One can use semi-empirical formulas to calculate finite heterogeneous buildup factors. Because of the approximations made in their derivations, the applications of these buildup factor formulas are limited in scope. It is generally agreed that there is a lack of information about finite homogeneous (single layer) and finite heterogeneous (multilayer) buildup factors. The transmission matrix method is believed to be able to furnish some of this formation.

The objective of this investigation is to apply the transmission matrix method for calculations of single layer and multilayer buildup factors. Shielding materials that will be used to form these layers are: water, aluminum, iron, lead and uranium. For purpose of comparison with other methods, the maximum thickness studied was 20 mean-free-path lengths. For the cases of infinite homogeneous media, buildup factors calculated by transmission matrix method were compared with the moments method results. For the cases of finite multilayer buildup factors, comparisons of the transmission matrix method results were made with those of the formulas of Broder, Kitazume and Kalos and the method of analytical continuation.

## II. REVIEW OF LITERATURE

### A. Methods of Evaluating Single Layer Gamma Ray Buildup Factors

A general survey of gamma ray buildup factors was treated by Trubey [1] and Chilton [2]. Systematic evaluation of gamma ray buildup factors was first done by Goldstein and Wilkins [3]. Using the moments method, buildup factors for infinite, homogeneous medium of water, aluminum, iron, tin, tungsten, lead and uranium were computed for point isotropic and plane monodirectional sources. These buildup factors of infinite, homogeneous medium have served as the benchmark values. Perkins [4] and Zerby [5] calculated finite energy (fluence) and exposure buildup factors for water, beryllium, aluminum, iron, tin and lead by using the Monte Carlo methods. Berger and Dogget [6] calculated the ratio of finite to semi-infinite buildup factors,

$$\frac{B_E\ (X,\ X)\ -1}{B_E\ (X,\ \infty)\ -1},$$

also by using the Monte Carlo method.

The simplest analytical approximation to Goldstein and Wilkins' result is by the empirical linear form [1]. The linear form is a very crude approximation and limited to short distance only. Berger [7] formulated an empirical, two-parameter exponential expression to fit Goldstein and Wilkins' results. Chilton [8] obtained values of these two parameters. The values of buildup factors from Berger form agree with Goldstein and Wilkins' results within error limits of the original data out to 10 mean-free-path lengths. Taylor [9] approximated Goldstein and Wilkins' results by formulating an empirical expression with two exponential terms. The accuracy of Taylor's form is not always as good as some other methods but the relation is easy to apply to physical problems. Capo's polynomial form [10] provides good agreement with Goldstein and Wilkins' results. However, the nature of polynomial expression makes it difficult to be applied to physical problems. All the empirical forms that are used to approximate the Goldstein and Wilkins' results make the assumption that the medium is infinite and homogeneous. There have not been established benchmark values of buildup factors for shield media of finite thicknesses.

B. Methods of Evaluating Multilayer Gamma Ray Buildup Factors

Several crude estimates of multilayer buildup factors are known. Among them are the methods of Rockwell of "conversion into equivalent layer thicknesses" and "dominance of high Z materials" [11]. Blizzard [12] suggested the method of "dominance of the last layer." Goldstein [13] proposed homogenization of the shielding layers by means of an homogenized effective atomic number. A more refined formula has been given by Broder [14] where the buildup at each layer is assumed to be the sum of individual differences in buildup. Kitazume modified Broder's form by multiplying each term in Broder's form by an exponentially decaying function [15,16]. This exponentially decaying function describes the final saturating buildup in the last layer. Kalos [17] devised a semiempirical formula to fit his Monte Carlo calculation of water-lead two layer buildup factors. Strictly speaking, Kalos' formula is valid only for a monodirectional source through a water and lead combination of layer thickness not greater than three mean-free-path lengths. Zumach [18] reasoned that the dose at the shield interfacial boundary must be continuous and hence devised the method of analytical continuation. This method gives results similar to those of Broder's formula. All these methods are empirical or semi-empirical in nature. There have been no calculations of multilayer build-up factors without constraints in thickness and configuration.

## III. GENERAL THEORY

Transmission of gamma rays through a shielding medium can be conveniently described in terms of transmission without collision with the electrons of the medium material, the unscattered flux, and transmission in which there is at least one collision with an electron, the scattered flux. The buildup factor is the ratio of the total transmitted flux (scattered and unscattered) to the unscattered flux.

$$B(X,\Lambda,Z) = \frac{\text{total transmitted flux}}{\text{unscattered flux}} = \frac{\phi\ (X,\Lambda)}{\phi_u(X,\Lambda)}$$

$$= \frac{\phi_u(X,\Lambda) + \phi_s(X,\Lambda)}{\phi_u(X,\Lambda)} = 1 + \frac{\phi_s(X,\Lambda)}{\phi_u(X,\Lambda)} \qquad (1)$$

where $B(X,\Lambda,Z)$ is the buildup factor at source-detector distance X, gamma ray source wavelength $\Lambda$ and shield medium atomic no. Z.

$\phi(X,\Lambda)$, $\phi_u(X,\Lambda)$ and $\phi_s(X,\Lambda)$ are respectively the total transmitted flux, unscattered flux and scattered flux at source-detector distance X and gamma ray source wavelength $\Lambda$.

In view of the fact that the unscattered flux for a particular shield can be readily obtained, the task of computing buildup factors then becomes one of finding the total transmitted flux.

Gamma flux can be measured in either "energy fluence" ($Mev/cm^2$), "exposure" (Roentgen, R) or "energy deposition" (Rad)*. The buildup

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*See Appendix A for definitions.

factors with gamma flux measured in these units are called respectively "energy fluence", "exposure" and "energy deposition" buildup factors. These buildup factors were formerly known as "energy", "dose" and "energy absorption" buildup factors.

A. Calculation of Unscattered Gamma Flux--Lambert's Law

For a monoenergetic, collimated narrow beam (or monoenergetic, broad parallel rays in purely absorptive medium), the attentuation of gamma flux in the medium can be described by Lambert's Law:

$$\phi_u(X, \Lambda) = \phi(0, \Lambda) \exp(-\mu X) \tag{2}$$

where $\phi_u(X, \Lambda)$ = unscattered flux at position X, source wavelength $\Lambda$.

$\phi(0, \Lambda)$ = source flux at position 0, source wavelength $\Lambda$.

$\mu$ = linear attenuation coefficient of the attenuation medium.

$X$ = thickness of the attentuation medium.

For multilayer shields, the unscattered flux after the $n^{th}$ layer is:

$$\phi_u(X_N, \Lambda) = \phi(0, \Lambda) \prod_{i=1}^{n} \exp(-\mu_i X_i)$$

$$= \phi(0, \Lambda) \exp(-\sum_{i=1}^{n} \mu_i X_i) \tag{3}$$

where $\phi_u(X_N, \Lambda)$ = unscattered flux at $n^{th}$ layer, source wavelength $\Lambda$.

$\mu_i$ = linear attenuation coefficient of the $i^{th}$ layer

$X_i$ = thickness of the $i^{th}$ layer

$X_N$ = distance between source and the $n^{th}$ layer.

The linear attenuation coefficients of the various media and source wavelength are available in various publications, e.g. ref. [19]. Thus, the calculation of unscattered flux of a certain shield is simply the application of Equation (2) or Equation (3).

B. Calculation of Total Transmitted Gamma Flux--
Moments Method, Monte Carlo Method and Transmission Matrix Method

Insofar as buildup factor data are concerned, the calculation of total transmitted flux have been done mainly by moments method and to a lesser extent, by Monte Carlo methods. Both of these methods have inherent and sometimes serious limitations.

1. Moments method [20]

The moments method can be briefly described by the following:

First, expand the angular variable of the directional flux by Legendre polynomials and insert them back into the transport equation for photons. The angular dependence of the gamma flux is now expressed in terms of angular moments. Then expand the spatial variable of the gamma flux in a power series and integrate the whole transport equation over all space. Finally, one proceeds to solve for the gamma flux numerically by choosing a certain number of angular and spatial moments.

Since the number of angular and spatial moments can be chosen to be any number that is appropriate for the shield medium, the moments method can handle any degree of anisotropy and thickness. However, in implementing the process of integration over all space, the assumption of infinite medium was made. Furthermore, the integration is not feasible unless the cross section is spatially independent. Thus, one further assumes the medium to be homogeneous. Consequently, the total transmitted flux obtained by moments method is limited to single layer (i.e. homogeneous medium), and it includes the effect of source backing and reflection beyond the thickness of the layer.

2. Monte Carlo method

The Monte Carlo method is essentially an experiment using random numbers and can be very flexible in terms of geometry. A certain number of photons are emitted by the source and each one is traced through the medium using stochastic methods for reactions and particle transport. The total transmitted flux through the shield medium can accordingly be calculated by an aggregate of these random photon samplings. The shield medium can be homogeneous or heterogeneous with almost any geometric shape. In order to calculate the total transmitted flux within reasonable uncertainty, it is necessary to sample a large number of photons (2000 and upward). This requires large amounts of computer time and therefore is exceedingly expensive for detailed parametric studies for buildup factors.

3. Transmission matrix method [21]

Consider a slab shield of finite thickness $t_1$ with gamma ray incident on and leaving from both sides. The gamma flux transport problem can be described by the following expressions:

$$\phi_2 = T\,\phi_1 + R^*\Psi_2 \tag{4}$$

$$\Psi_1 = T^*\,\Psi_2 + R\,\phi_1 \tag{5}$$

where $\phi_1$ and $\Psi_1$ are respectively the incoming and outgoing flux on the left face of the slab, and $\phi_2$ and $\Psi_2$ are the outgoing and incoming flux on the right; $T_1$ and $T_1^*$ are respectively the transmission operators for flux incident on the left and right face, and $R_1$ and $R_1^*$ are the reflection operators for flux incident on the left and right face.

Excluding photon-photon reactions, there exists a linear operator H(t) such that

$$\begin{bmatrix} \phi_2 \\ \Psi_2 \end{bmatrix} = H(t) \begin{bmatrix} \phi_1 \\ \Psi_1 \end{bmatrix} . \qquad (6)$$

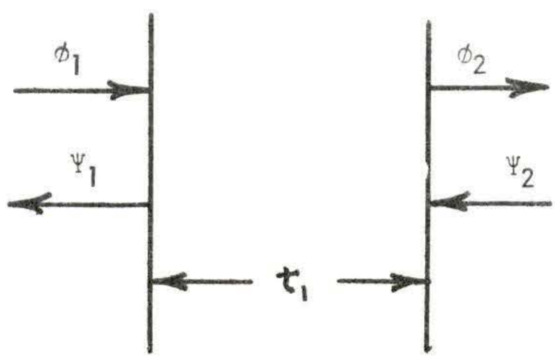



Figure 1. Slab shield

$\phi_2$ and $\Psi_2$ can be solved from Equations (4) and (5) in terms of $\phi_1$ and $\Psi_1$. From Equation (5), one obtains

$$\Psi_2 = T^{*-1} (\Psi_1 - R\phi_1) = -U^*R\, \phi_1 + U^*\Psi_1 . \qquad (7)$$

Substituting Equation (7) into Equation (4), one has the relation:

$$\phi_2 = T\phi_1 + R^*(-U^*R\phi_1 + U^*\Psi_1) = (T - R^*U^*R)\, \phi_1 + R^*U^*\Psi_1 . \qquad (8)$$

where $U = T^{-1}$, $U^{*} = T^{*-1}$. For isotropic materials, $T = T^{*}$ and $R = R^{*}$.

Combining Equations (7) and (8) into a matrix equation, one obtains:

$$\begin{bmatrix} \phi_2 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} T-R^{*}U^{*}R & R^{*}U^{*} \\ -U^{*}R & U^{*} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \Psi_1 \end{bmatrix} \tag{9}$$

Comparing Equations (6) and (9), the H operator becomes

$$H = \begin{bmatrix} T - R^{*}U^{*}R & R^{*}U^{*} \\ -U^{*}R & U^{*} \end{bmatrix} \tag{10}$$

For a two-layer slab shield, consider the following figure:

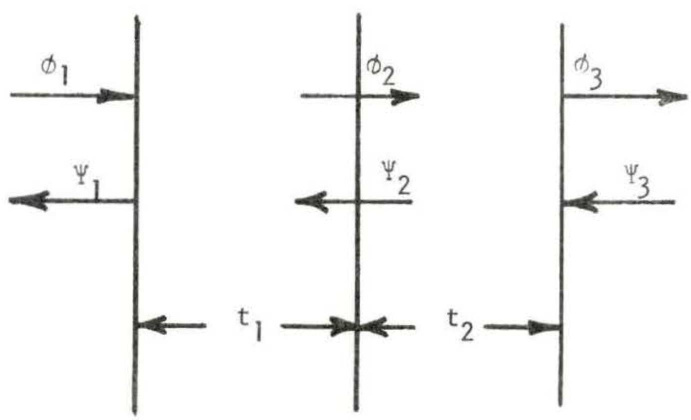



Figure 2. Two-layer slab shield

One can write for the first layer:

$$\begin{bmatrix} \phi_2 \\ \Psi_2 \end{bmatrix} = H(t_1) \begin{bmatrix} \phi_1 \\ \Psi_1 \end{bmatrix} = \begin{bmatrix} T_1-R_1^{*}U_1^{*}R_1 & R_1^{*}U_1^{*} \\ -U_1^{*}R_1 & U_1^{*} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \Psi_1 \end{bmatrix} \tag{11}$$

and the second layer:

$$\begin{bmatrix}\phi_3\\ \Psi_3\end{bmatrix} = H(t_2)\begin{bmatrix}\phi_2\\ \Psi_2\end{bmatrix} = \begin{bmatrix} T_2 - R_2^{*} U_2^{*} R_2 & R_2^{*} U_2^{*} \\ -U_2^{*} R_2 & U_2^{*}\end{bmatrix}\begin{bmatrix}\phi_2\\ \Psi_2\end{bmatrix} \tag{12}$$

Combining Equations (11) and (12), one obtains:

$$\begin{bmatrix}\phi_3\\ \Psi_3\end{bmatrix} = H(t)\begin{bmatrix}\phi_1\\ \Psi_1\end{bmatrix} = \begin{bmatrix} T - R^{*} U^{*} R & R^{*} U^{*} \\ -U^{*} R & U^{*}\end{bmatrix}\begin{bmatrix}\phi_1\\ \Psi_1\end{bmatrix} \tag{13}$$

$$\text{where } T = T_2[I - R_1^{*}R_2]^{-1}T_1 = T_2 \sum_{n=o}^{\infty} (R_1^{*}R_2)^n\, T_1 \,. \tag{14}$$

$$R = R_1 + T_1^{*}(I - R_2R_1^{*})^{-1}R_2T_1$$

$$= R_1 + T_1^{*} R_2 \sum_{n=o}^{\infty} (R_1^{*}R_2)^n\, T_1 \tag{15}$$

$$T^{*} = T_1^{*}\, [I - R_2R_1^{*}]^{-1}\, T_2^{*}$$

$$= T_1^{*} \sum_{n=o}^{\infty} (R_2R_1^{*})^n\, T_2^{*} \tag{16}$$

$$R^{*} = R_2^{*} + T_2R_2^{*}\, [I - R_2R_1^{*}]^{-1}\, T_2^{*}$$

$$= R_2^{*} + T_2R_2^{*} \sum_{n=o}^{\infty} (R_2R_1^{*})^n\, T_2^{*} \tag{17}$$

The derivation for a n-layer slab shield follows the same logic as in the two-layer case. From Equations (9) and (13) through (17), it is observed that in order to obtain the total transmitted flux, one must find the transmission and reflection matrix operators, T(t) and R(t). Detailed derivations for finding these two matrix operators are given in Appendix B. With T(t) and R(t) obtained, the total transmitted flux (i.e. $[{}^{\phi_2}_{\Psi_2}]$ for a single layer and $[{}^{\phi_n}_{\Psi_n}]$ for a n-layer shield) can be computed accordingly. The transmission matrix method is an analytical process which does not assume the properties of infinite homogeneous medium.

C. Description of Buildup Factor Formulas

In this section, various analytical approximations used to describe buildup factor data are examined.

1. Single layer buildup factors

The bulk of single layer buildup factors have been calculated by the moments method [3,22]. These buildup factors are applicable to infinite homogeneous medium only due to the inherent properties of moments method. Goldstein and Wilkins [3] published an extensive table of these buildup factors. Uncertainties claimed in these calculations were from 5 to 10 percent. Various efforts have been made to approximate these buildup factors by analytical formulas. They, therefore, are for an infinite homogeneous medium also. The four most commonly used formulas will be described here.

(a) Linear form: $B(\Lambda,\mu X) = 1 + A_1(\Lambda)\mu X$ (18)

where $\Lambda$ = source wavelength

$\mu$ = linear attenuation coefficient

X = source-detector distance

$A_1(\Lambda)$ = constant = $B(\Lambda,1)-1$ .

The linear form was established by reasoning that the scattered buildup,

$B(\Lambda, \mu X)-1$, varies linearly with the distance. This form does not take into account the saturation in buildup at deep penetrations. In general, the linear form gives a higher value than the Goldstein and Wilkins' results and is valid only at short distance, usually on the order of a mean-free-path length. The value of the constant $A_1(\Lambda)$, can be readily obtained by setting the distance at one mean-free-path length.

(b) Berger form: $B(\Lambda, \mu X) = 1 + C(\Lambda)\mu X e^{D(\Lambda)\mu X}$ (19)

where $C(\Lambda)$ and $D(\Lambda)$ are the two parameters dependent on the source wavelength.

Having observed the buildup factors displayed saturation as the distance increases, Berger [7] reasoned that the scattered buildup, $B(\Lambda, \mu X)-1$, should follow an exponential increase. The Berger form results match well with Goldstein and Wilkins' result over the range out to 10 mean-free-path lengths. Because of its exponential expression, the Berger form is quite conveniently applied to integration to form the plane isotropic source kernel. Note that this formula reduces to unity when the distance is zero. The two parameters, $C(\Lambda)$ and $D(\Lambda)$, are available in ref. [8].

(c) Taylor form:

$$B(\Lambda, \mu X) = Ae^{-\alpha_1(\Lambda)\mu X} + (1 - A)e^{-\alpha_2(\Lambda)\mu X} \qquad (20)$$

where A, $\alpha_1(\Lambda)$ and $\alpha_2(\Lambda)$ are parameters to be determined for different source wavelengths.

Taylor pointed out the spatial dependence of buildup factors can be approximated by the sum of two exponential terms. In essence, this form stated that the buildup is the sum of an exponential scattered term and the exponential of an unscattered term. Taylor's form also reduces to unity at zero thickness but its accuracy is not always comparable to the original Goldstein and Wilkins' data. However, because of its exponential expression, this formula is frequently applied to integration for conversion to plane isotropic source kernel. The value of the three parameters are available in ref. [10].

(d) Polynomial form of Capo:

$$B(\Lambda,\mu X) = \sum_{n=o}^{3} \beta_n(\Lambda)(\mu X)^n \qquad (21)$$

where $\beta_n(\Lambda)$ are the polynomial coefficients.

Unlike the above three formulas, Capo's form utilized a 4-term polynomial to approximate the Goldstein and Wilkins' data. This formula is the most accurate one among the analytical approximation formulas. It matches the Goldstein and Wilkins' buildup factor very closely over the whole range of distances (out to 15 or 20 mfp) and for energies from 0.5 to 10 Mev. However, it is rather difficult to apply this form to integration to form the plane source kernel. The complete set of polynomial coefficients are given in ref. [9].

Monte Carlo methods were used in calculating buildup factors of water, beryllium, aluminum, iron, antimony and lead of only a few mean-free-path lengths because of the extensive computer time this method requires. No complete tabulation of buildup factors as a function of source energy, thickness and materials have been done.

By applying the algorithm of the transmission matrix method, systematic compilation of buildup factors was done for water, aluminum, iron, lead and uranium. The transmission matrix method can handle both the infinite medium and finite slab geometries. With a slight modification, it can also treat the geometric configurations of semi-infinite medium with source backing and semi-infinite medium with reflection. Comparison with Goldstein and Wilkins' result was made for the infinite medium calculations.

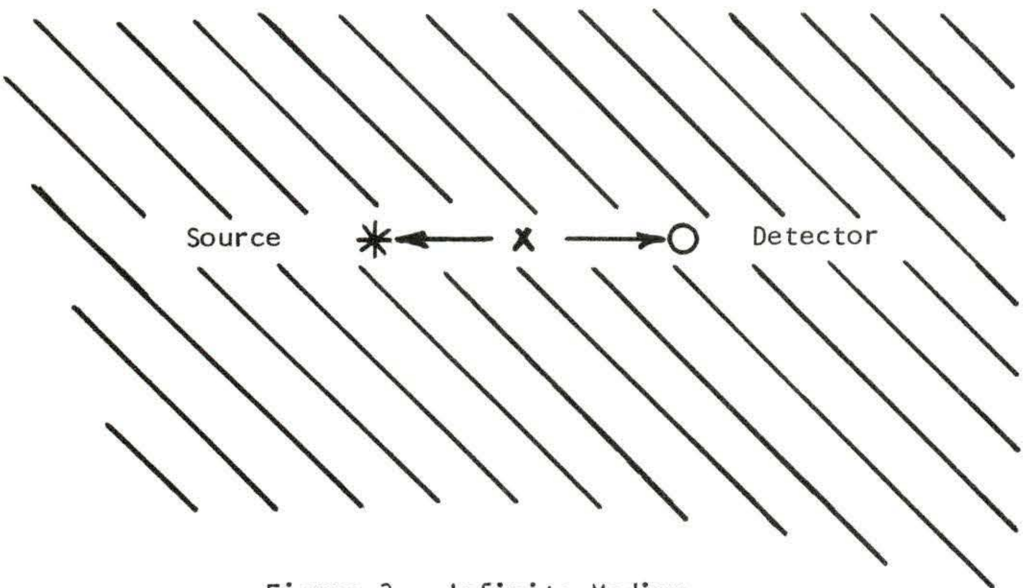



Figure 3. Infinite Medium

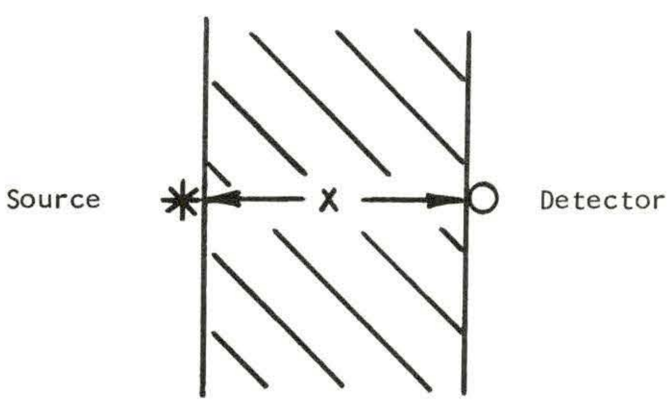



Figure 4. Finite Slab

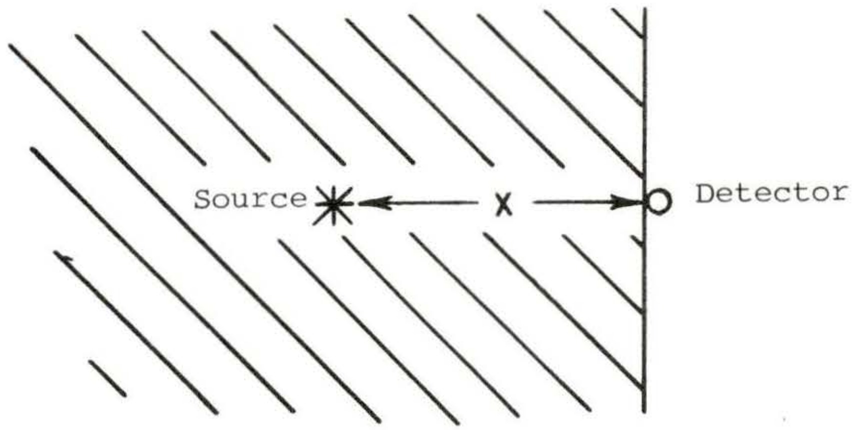



Figure 5. Semi-infinite medium with source backing

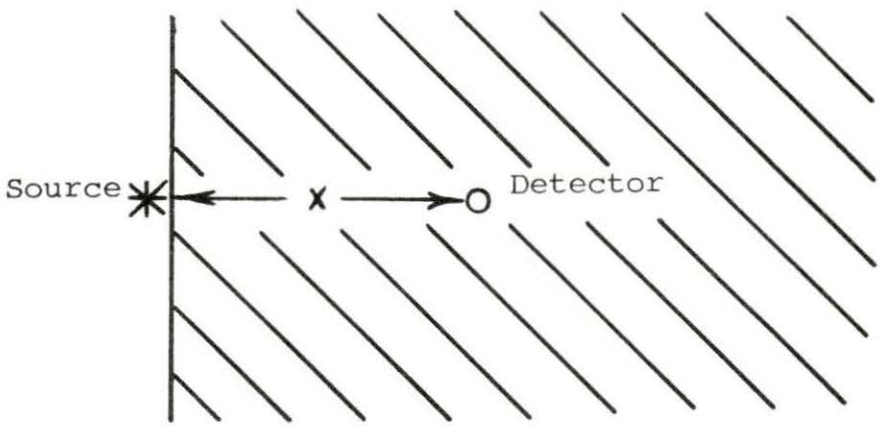



Figure 6. Semi-infinite medium with reflection

2. Multilayer buildup factors

Because of the inability of the moments method to treat analytically gamma transport at the interfacial boundary of a multilayer shield, the development of multilayer buildup factors are semiempirical at best. Among the various proposals, the following four are commonly used:

(a) Broder's Formula  Based on shielding experiments using a point isotropic source of cobalt-60 and laminated layers of polyethylene, aluminum, iron and lead, Broder et al. [14] stated that the multilayer buildup factors are governed by:

$$B\left(\sum_{i=1}^{N} X_i\right) = \sum_{n=1}^{N} B_n\left(\sum_{i=1}^{N} X_i\right) - \sum_{n=2}^{N} B_n\left(\sum_{i=1}^{n-1} X_i\right) \qquad (22)$$

where $X_i$ = thickness of the i-th layer in mean-free-path.

For a 2-layer system, N = 2, the expression becomes:

$$B(X_1 + X_2) = B_2(X_1 + X_2) + [B_1(X_1) - B_2(X_1)] \; . \qquad (23)$$

For a 3-layer system, N = 3, the expression becomes:

$$B(X_1 + X_2 + X_3) = B_3(X_1 + X_2 + X_3) + [B_1(X_1) - B_2(X_1)]$$

$$+ [B_2(X_1 + X_2) - B_3(X_1 + X_2)] \; . \qquad (24)$$

In essence, the Broder's Formula stated that the total buildup factor of a multilayer system is the sum of the individual differences in the buildup. This relation showed good agreement with experimental results for heavy-light systems (e.g., lead followed by aluminum), but was found to be inadequate for light-heavy systems (e.g., water followed by lead). It should be noted that the shielding experiments were not performed with a monoenergetic source. Total experimental error claimed was ± 10%. Application of Broder's formula, therefore, can only be used as a rule-of-thumb estimation.

(b) Kitazume's formula  Based on the results of the shielding experiments using a plane source of cobalt-60 (activity: 10 Ci) and laminated layers of water, iron and lead, Kitazume modified Broder's formula by multiplying each term by an exponentially decaying function, $\exp(-\alpha X_r)$, where $X_r$ is the distance to the end point measured in mean-free-path [15,16].

Thus, for a N-layer system, Kitazume's formula is:

$$B\left(\sum_{i=1}^{N} X_i\right) = \sum_{n=1}^{N} B_n\left(\sum_{i=1}^{n} X_i\right) \exp\left(-\alpha \sum_{r=n+1}^{N} X_r\right) - \sum_{n=2}^{N} B_n\left(\sum_{i=1}^{n-1} X_i\right) \exp\left(-\alpha \sum_{r=n}^{N} X_r\right) \tag{25}$$

For a two-layer system, N = 2, the expression becomes:

$$B(X_1 + X_2) = B_2(X_1 + X_2) + [B_1(X_1) - B_2(X_1)] \exp(-\alpha X_2) \tag{26}$$

For a three-layer system, N = 3, the expression becomes:

$$B(X_1 + X_2 + X_3) = B_3(X_1 + X_2 + X_3) + [B_1(X_1) - B_2(X_1)] \exp[-\alpha(X_2 + X_3)] + [B_2(X_1 + X_2) - B_3(X_1 + X_2)] \exp[-\alpha(X_3)] \tag{27}$$

Equation (26) is for low energy (1-2 Mev) gamma. For high energy gamma

($E > 2$ Mev), the following version gives better results:

$$B(X_1 + X_2) = B_1(X_1 + X_2) + [B_1(X_1) - B_2(X_1)] \exp(-\alpha X_2)$$

$$+ A[B_2(X_1 + X_2) - B_2(X_2)] \exp(-\beta/X_2) \qquad (28)$$

where A, $\alpha$ and $\beta$ are determined experimentally.

The introduction of the exponential term is designed to take into account the final saturation buildup in the last layer. Hence, the determination of $\alpha$ is critical. This can be done experimentally or by direct numerical integration. In general, $\alpha$ is between 0 and 3. For heavy-light combinations, Kitazume form reduces to Broder form by setting $\alpha = 0$. For light-heavy combinations, $\alpha \simeq 3$. Kitazume form suffers also from uncertainties due to experimental errors. Deviations of Kitazume's formula from experimental values were $\pm$ 5-15%. Thus, the results of Kitazume's formula can only be used as an approximation to the multilayer buildup factors.

(c) Kalos' formula for lead and water shield   Gamma transport studies of 3 mfp layers of water and lead were done by Kalos using Monte Carlo techniques. Buildup factors were obtained for these systems and the following semi-empirical formulas were devised to fit the results:

For lead followed by water; $0.5$ Mev $\leq E \leq 10$ Mev;

$$B(X_1 + X_2) = B_2(X_2) + \frac{B_1(X_1)-1}{B_2(X_1)-1} [B_2(X_1 + X_2)-B_2(X_2)] \qquad (29)$$

For water followed by lead and $0.5$ Mev $\leq E \leq 10$ Mev

$$B(X_1 + X_2) = B_2(X_2) + \left[ \frac{B_1(X_1)-1}{B_2(X_1)-1} \exp(-1.7\, X_2) + \frac{(\mu_c/\mu_t)_1}{(\mu_c/\mu_t)_2} (1-\exp(-X_2))\right]$$

$$[B_2(X_1 + X_2) - B_2(X_2)] \qquad (30)$$

where $(\mu_c/\mu_t)_n$ is the ratio of Compton scattering to total cross section in the n-th material.

Efforts have been made [16,17] to extend Kalos' formula to other materials and over 3 mfp in layer thickness. In general, the deviation increases as layer thickness increases beyond 3 mfp. Application of Kalos' formula is therefore limited to heavy-light thin shield systems.

(d) The method of "analytical continuation" The method of "analytical continuation" was proposed by Zumach based on the following two conditions:

For a N-layer system with individual layer thickness $X_1$, $X_2$, $X_3 \ldots X_N$,

(i) The dose at the n-th to n+1-th interfacial boundary must be continuous.

(ii) The multilayer buildup factor at a thick layer approaches the material buildup of this layer.

$$B(X) \rightrightarrows B_N(X) \quad \text{for } X_N \gg 1$$

where B(X) is the multilayer buildup factor with total thickness X mfp.

$B_N(X)$ is the material buildup factor for the N-th layer with thickness X mfp

$X_N$ is the thickness of the N-th layer.

Consider that the first layer buildup factor, $B_1(X_1)$ is known, then one can select a mfp $Y_1$ from the second layer such that:

$$B_1(X_1) = B_2(Y_1) \quad . \tag{31}$$

The two-layer buildup factor at the second layer is therefore:

$$B(X_1 + X_2) = B_2 \ (Y_1 + X_2) \ . \tag{32}$$

Carrying this procedure out to the N-th layer, one obtains:

$$B(\sum_{i=1}^{N} X_i) = B_N \ (Y_{N-1} + X_N) \tag{33}$$

It is observed that the above procedures are justified if and only if the energy spectrum and angular dependence does not change appreciably while transversing from one layer to the adjacent one. This restriction is evidently violated for a heavy-light combination. Thus, the method of analytical continuation is limited to systems consisting of layers of similar attenuating and scattering properties.

The transmission matrix method is one of the analytical methods that can compute multilayer buildup factors with no constraints on shield materials and layer thickness. The procedure is similar to that of single layer buildup factors as described by Appendix B. When Equations (3) and (14) - (17) were solved, the buildup factors were automatically obtained by taking the ratio of total transmitted flux to unscattered flux.

The results from the transmission matrix method will be discussed in the next section.

# IV. RESULTS AND DISCUSSION

## A. Single Layer Buildup Factor Studies

The transmission matrix method was employed to compute single layer buildup factors of water, aluminum, iron, lead and uranium for infinite medium and finite slab geometry. It was observed that the infinite medium buildup factors agreed quite well with the Goldstein and Wilkins result. The comparisons of these buildup factors are shown in Figures 7 - 11. Because of the close agreements displayed in the comparisons, the technique of buildup factor evaluation by transmission matrix method is believed to be correct within the context of this investigation.

For light (low Z) materials, the effects of source backing and reflection beyond finite thicknesses cause the infinite medium buildup factors to be considerably higher than those of finite slabs. The finite slab buildup factors are shown as the triangular points in Figures 7 - 11. As the atomic number of the material increases, the difference between infinite medium and finite slab decreases. For lead and uranium, the effects of source backing and reflection beyond finite thickness are almost negligible.

## B. Multilayer Buildup Factor Studies

The transmission matrix method was employed to compute multilayer buildup factors for twenty-six different two-layer and three-layer systems. Finite slab energy fluence buildup factors obtained from part A were used as input data to compute multilayer buildup factors by Broder's, Kitazume's, Kalos' formulas and the method of analytical continuation. Comparisons

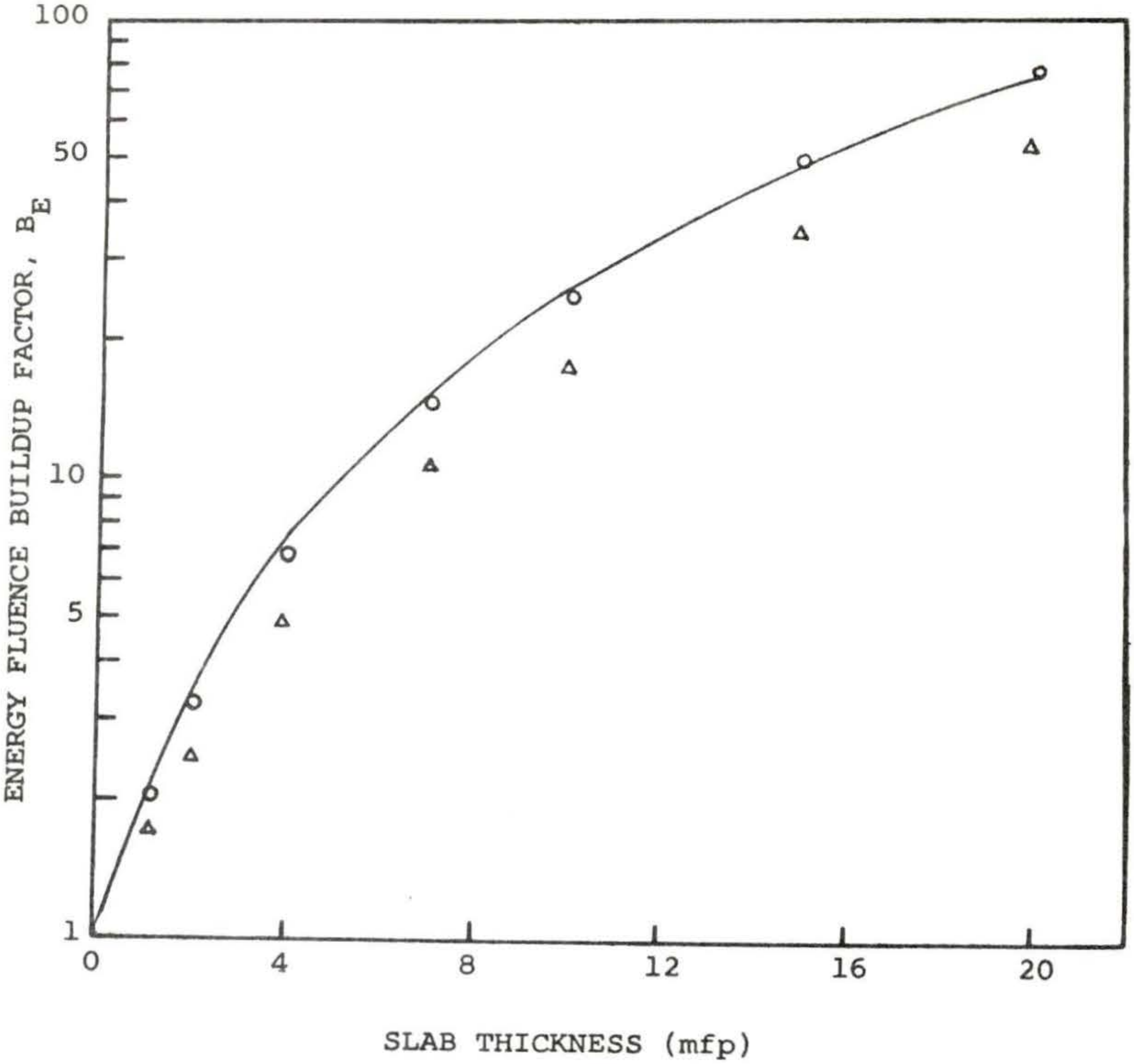



Figure 7. Energy fluence buildup factor, $B_E$, of water for a 1 Mev point isotropic source

———— : Moments Method calculations ( NYO-3075 )

o o o o : Transmission Matrix Method calculations, infinite medium

△ △ △ △ : Transmission Matrix Method calculations, finite slab

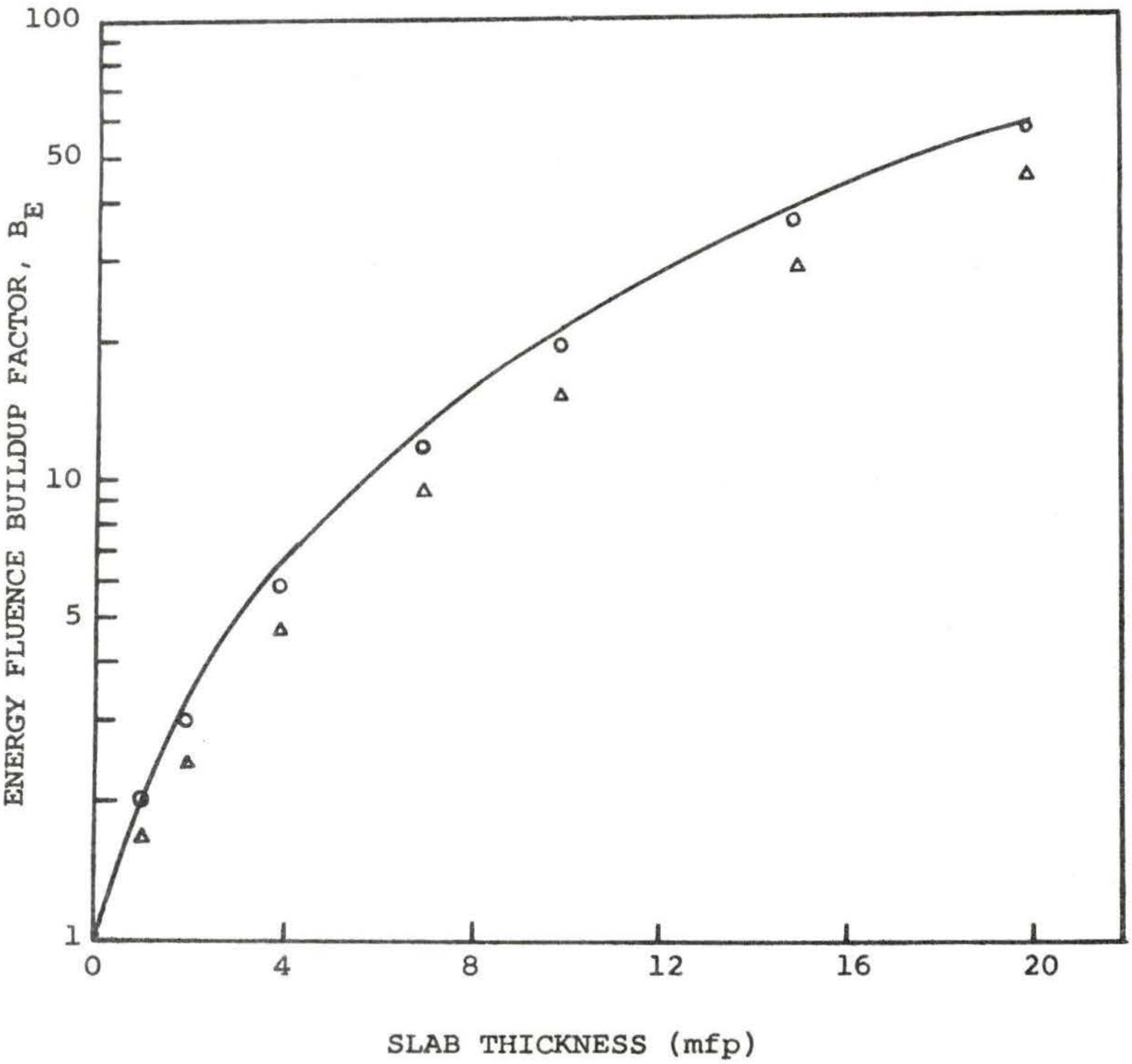



Figure 8. Energy fluence buildup factor, $B_E$, of aluminum for a 1 Mev point isotropic source

———— : Moments Method calculations ( NYO-3075 )

o o o o : Transmission Matrix Method calculations, infinite medium

Δ Δ Δ Δ : Transmission Matrix Method calculations, finite slab

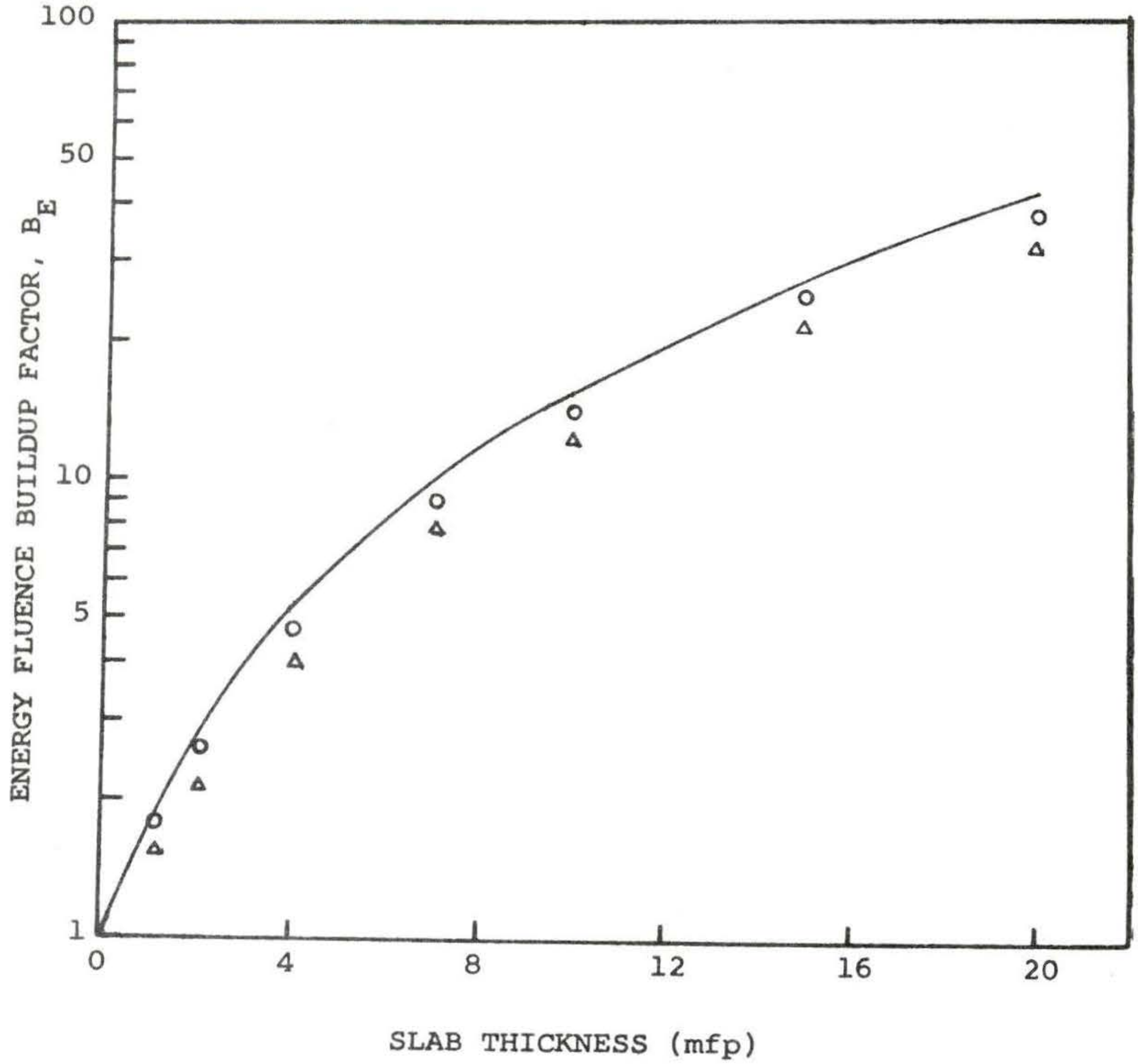



Figure 9. Energy fluence buildup factor, $B_E$, of iron for a 1 Mev point isotropic source

———————: Moments Method calculations ( NYO-30-75 )

o o o o : Transmission Matrix Method calculations, infinite medium

Δ Δ Δ Δ : Transmission Matrix Method calculations, finite slab

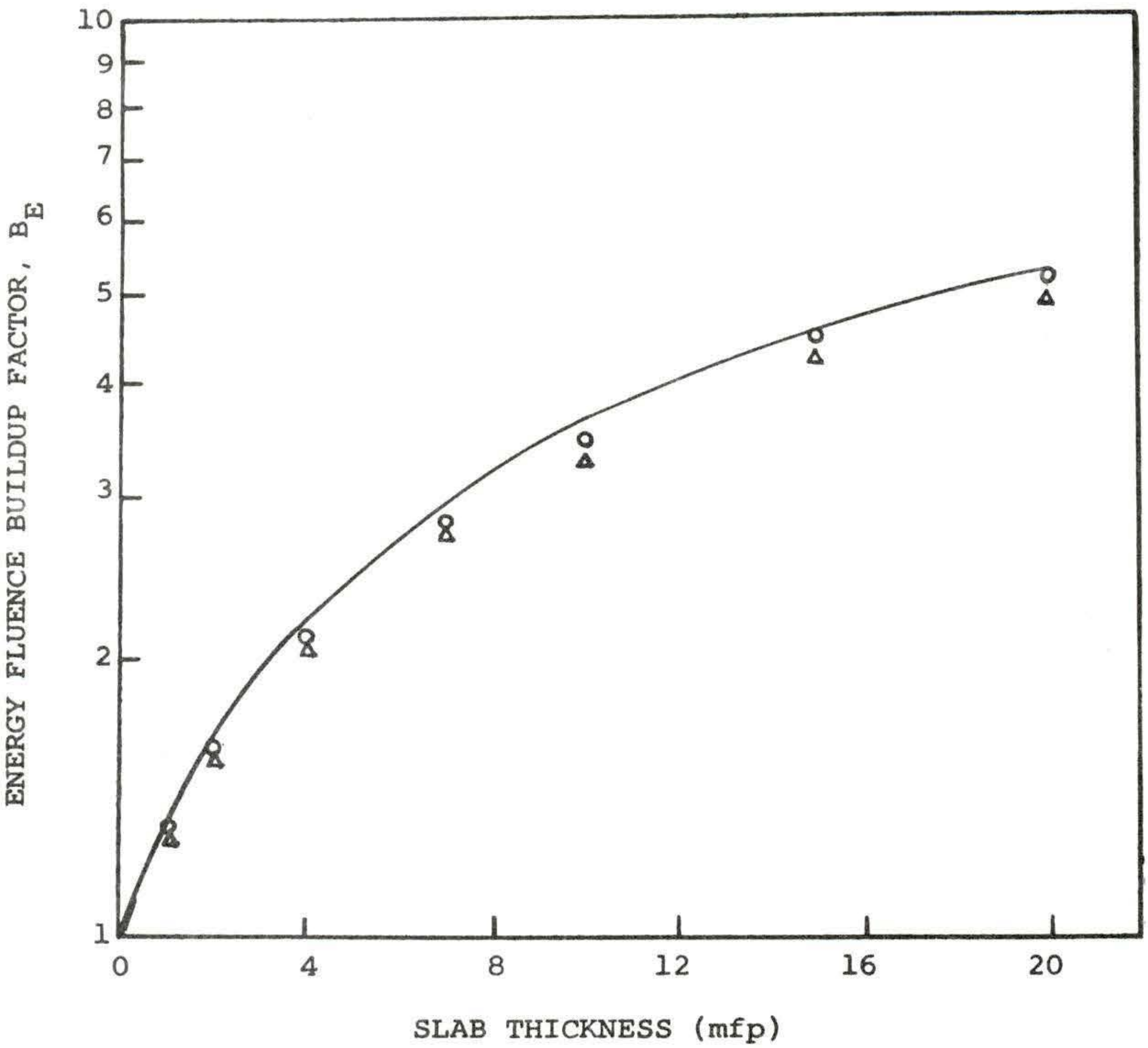



Figure 10. Energy fluence buildup factor, $B_E$, of lead for a 1 Mev point isotropic source

———— : Moments Method calculations ( NYO-3075 )

o o o : Transmission Matrix Method calculations, infinite medium

Δ Δ Δ : Transmission Matrix Method calculations, finite slab

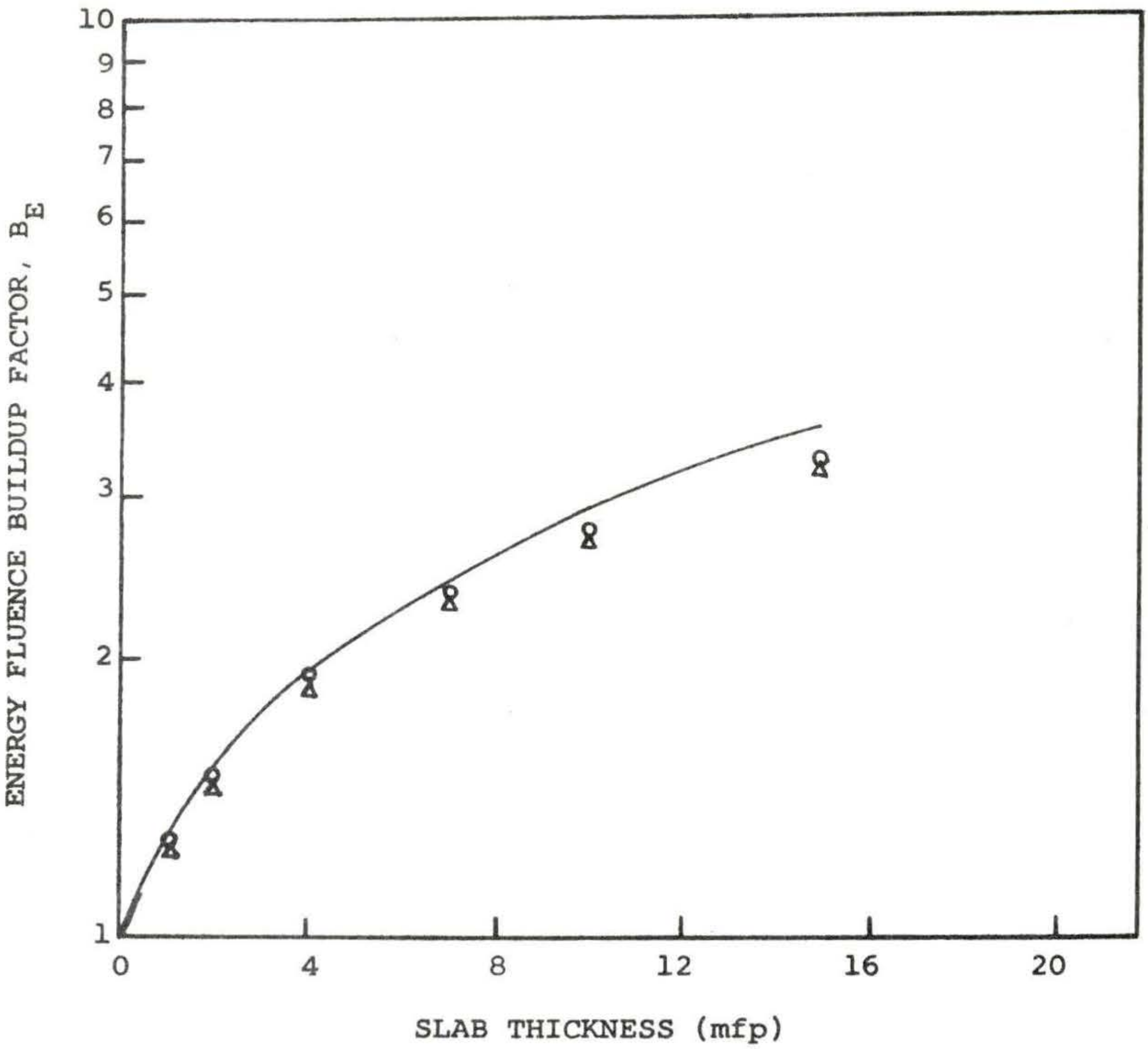



Figure 11. Energy fluence buildup factor, $B_E$, of uranium for a 1 Mev point isotropic source

——————— : Moments Method calculations ( NYO-3075 )

o o o : Transmission Matrix Method calculations, infinite medium

Δ Δ Δ : Transmission Matrix Method calculations, finite slab between the multilayer buildup factors computed by the transmission matrix method and those by the four semiempirical forms are shown in Figures 12 - 57. In the figures were displayed varying degrees of agreement between the transmission matrix method result and those of the four semi-empirical formulas. These differences will be discussed.

The various two-layer combinations can be classified into three broad categories:

1) Light material followed by heavy material, e.g., water followed by lead, iron followed by uranium, etc.

2) Heavy material followed by light material, e.g., lead followed by aluminum, uranium followed by water, etc.

3) Two layers of materials of similar properties such as the lead and uranium combinations.

The results obtained from this investigation will be discussed according to the above categories.

1. Light-heavy systems

It is apparent from Figures 12 - 51 that the formula of Broder is inadequate to describe a light-heavy system. The reason is that Broder's formula does not take into account the final saturating buildup in the last layer. In a light-heavy combination, this final saturating buildup in the last layer is dominated by the buildup of the last layer alone until the last layer thickness becomes relatively thin compared to the system thickness (approximately less than 5 mfp in a 20 mfp system and less than 1 mfp in a 6 mfp system). However, the Broder's form describes the buildup

Figure 12. Energy fluence buildup factor of a 20 mfp water-aluminum shield for a 1 Mev point isotropic source

Figures 12-57 are graphs of energy fluence buildup factor, $B_E$, versus first layer thickness of the multilayer slab shield systems for a 1 Mev point isotropic source. The media that compose the multilayer shields are stated in orders of appearances. For example, a uranium-water shield is a two-layer shield system with uranium as the first layer followed by water as the second layer. The total thicknesses of the multilayer shield systems are fixed at six or twenty mfp. The following abbreviations are used in the figures:

Transmission Matrix Method Calculations: TMMC

Broder's Formula Calculations: BFC

Kitazume's Formula Calculations: KFC

Kalos' Formula Calculations: KLFC

Method of Analytical Continuation Calculations: MACC

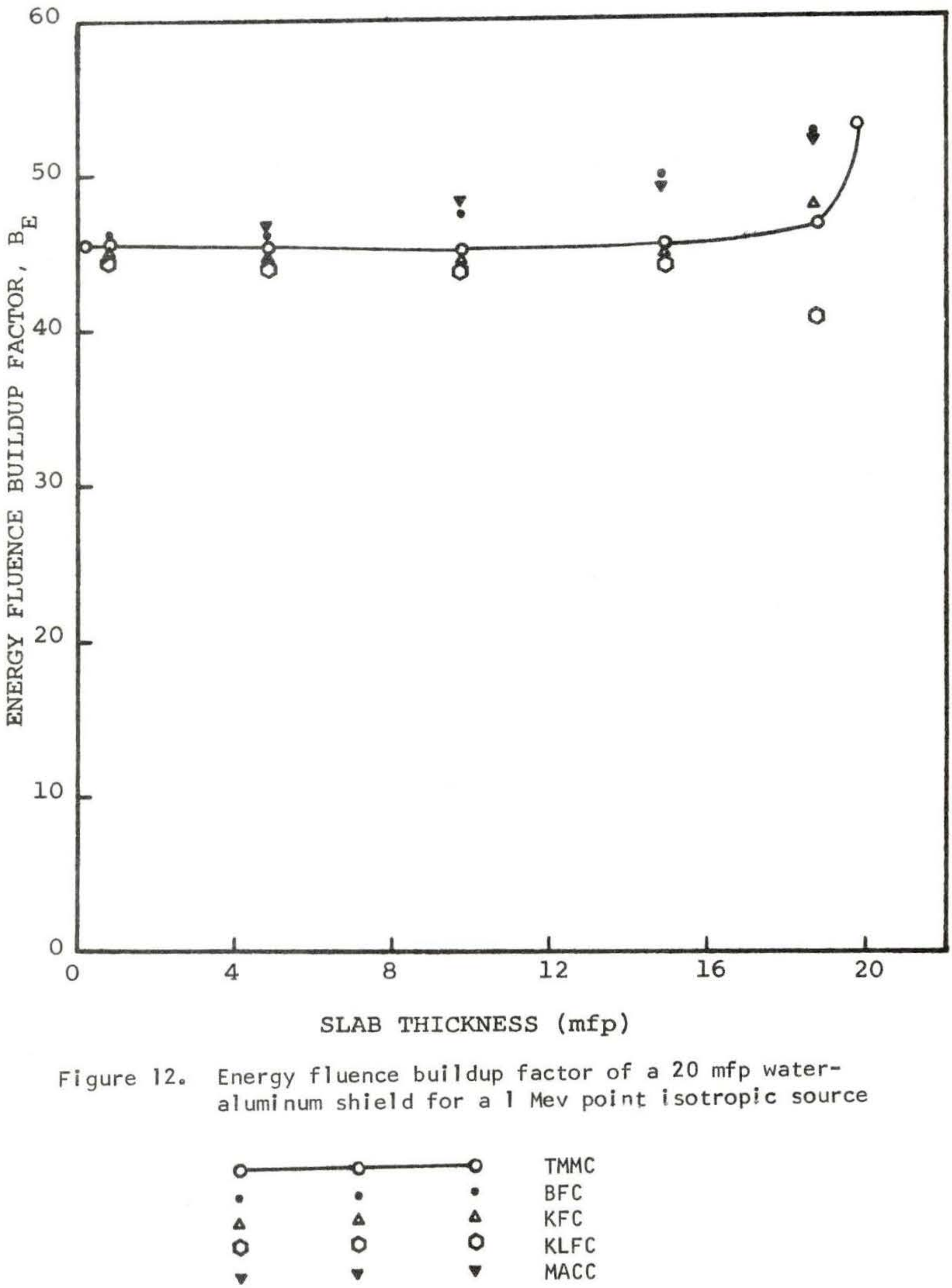



Figure 12. Energy fluence buildup factor of a 20 mfp water-aluminum shield for a 1 Mev point isotropic source

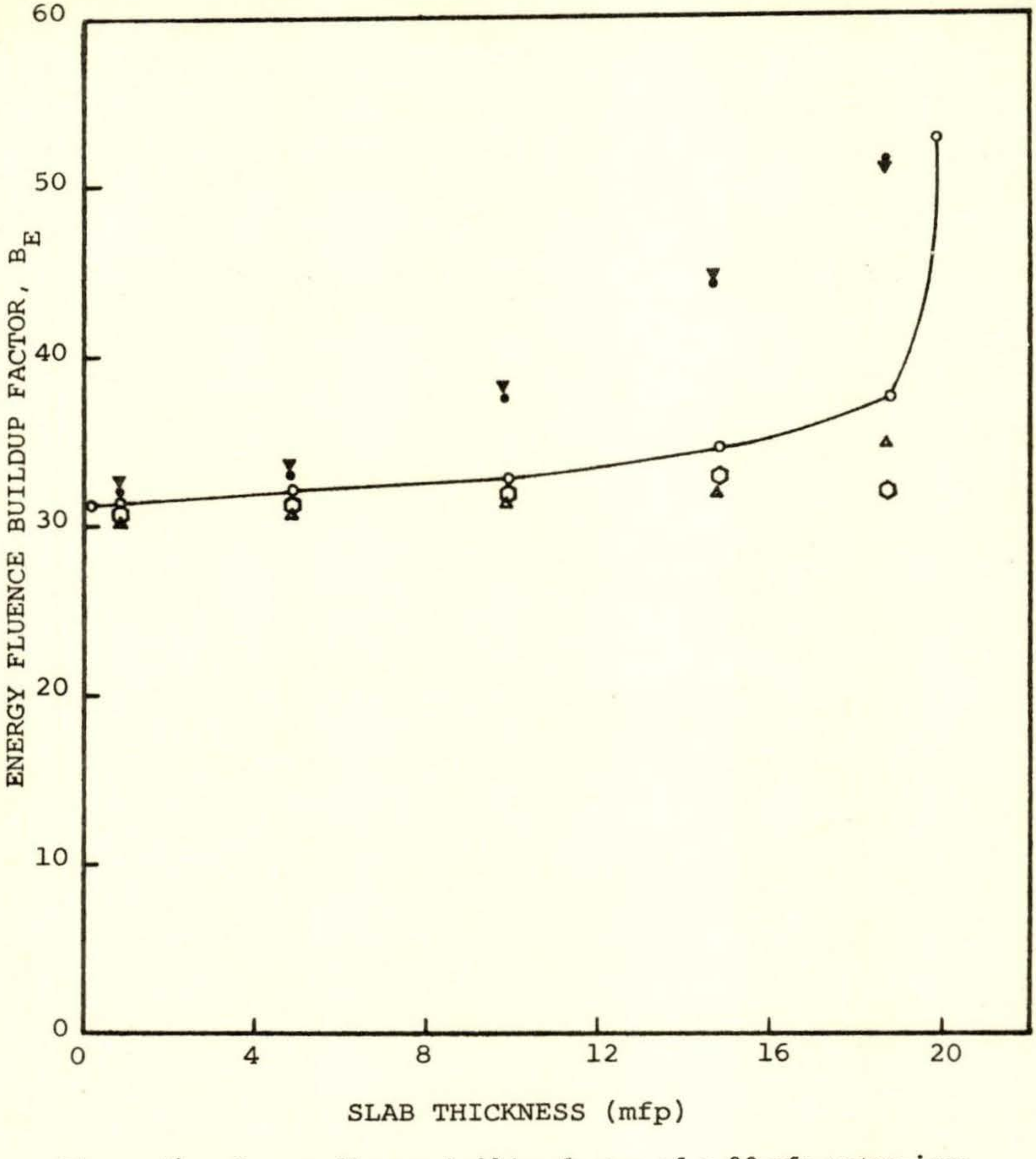
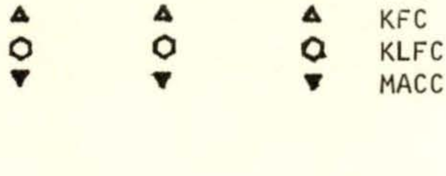



Figure 14. Energy fluence buildup factor of a 20 mfp water-iron shield for a 1 Mev point isotropic source



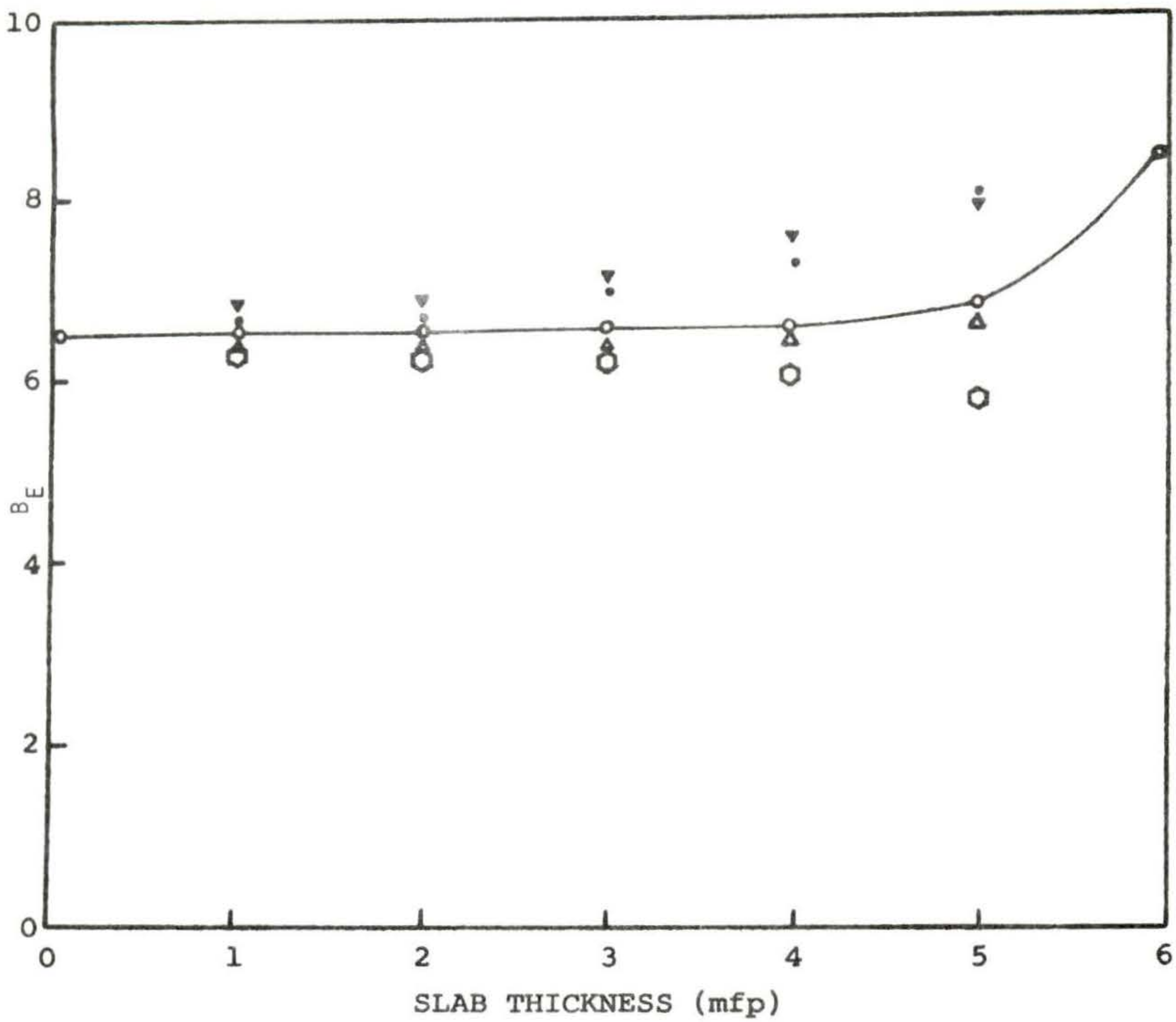
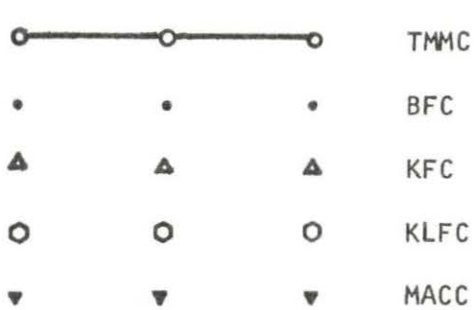



Figure 15. Energy fluence buildup factor of a 6 mfp water-iron shield for a 1 Mev point isotropic source



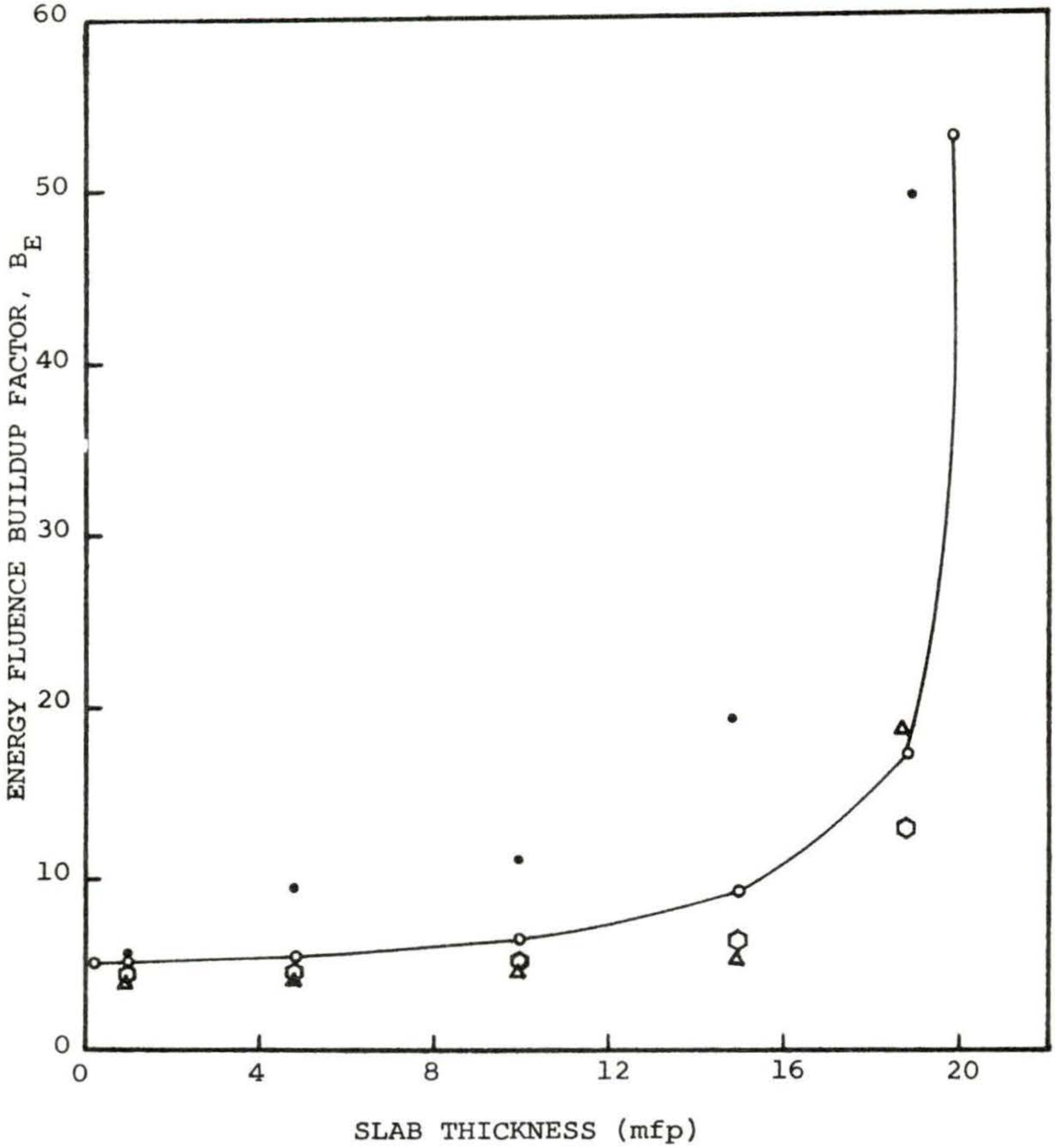
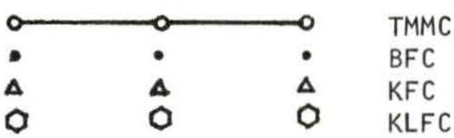



Figure 16. Energy fluence buildup factor of a 20 mfp water-lead shield for a 1 Mev point isotropic source



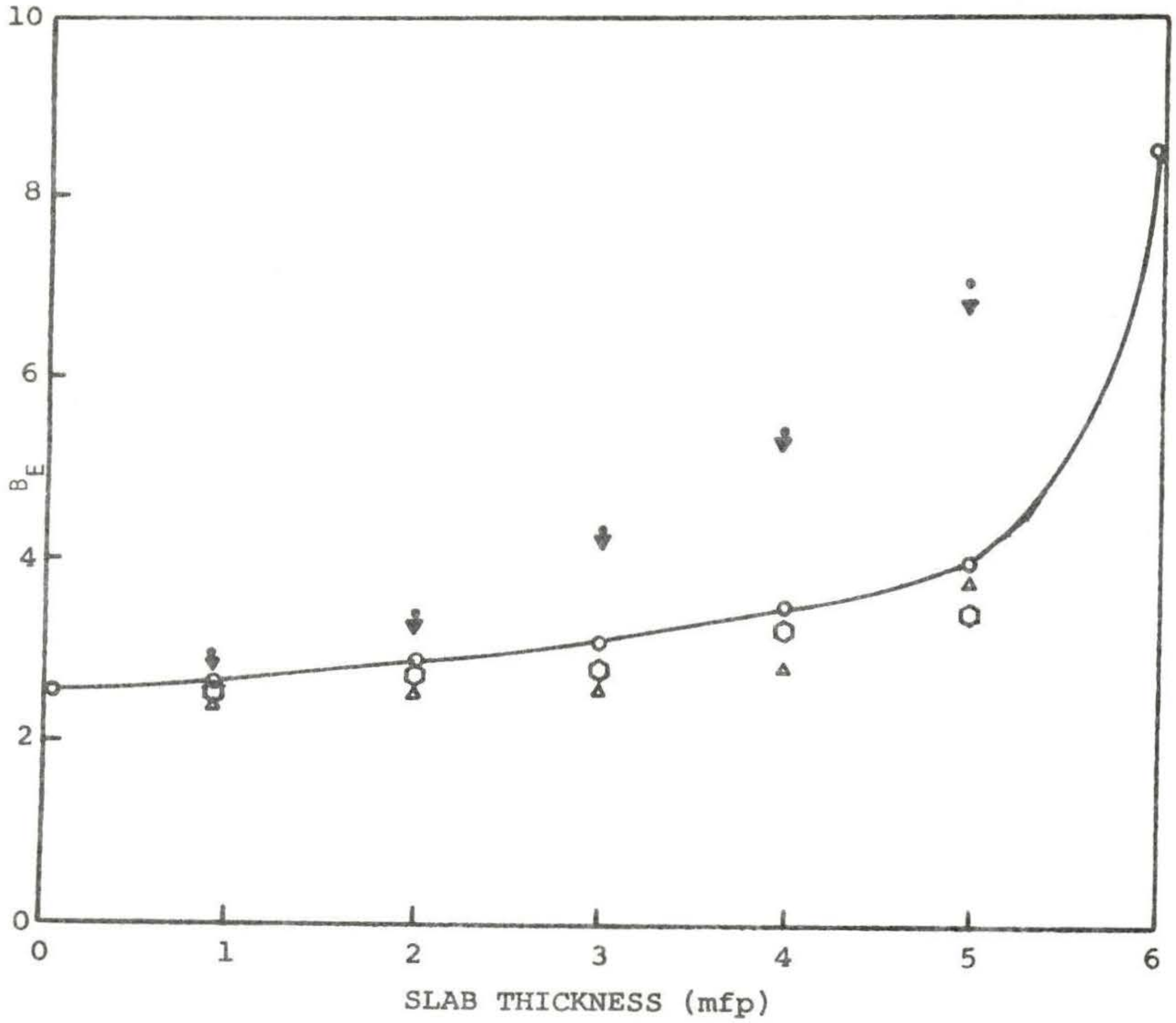
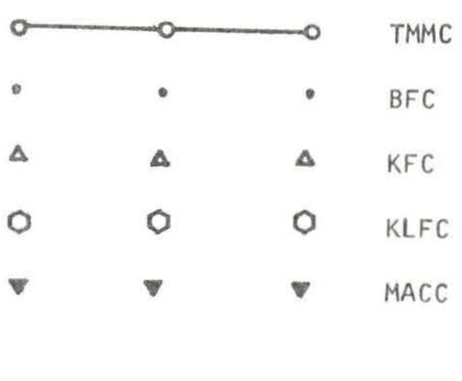



Figure 17. Energy fluence buildup factor of a 6 mfp water-lead shield for a 1 Mev point isotropic source



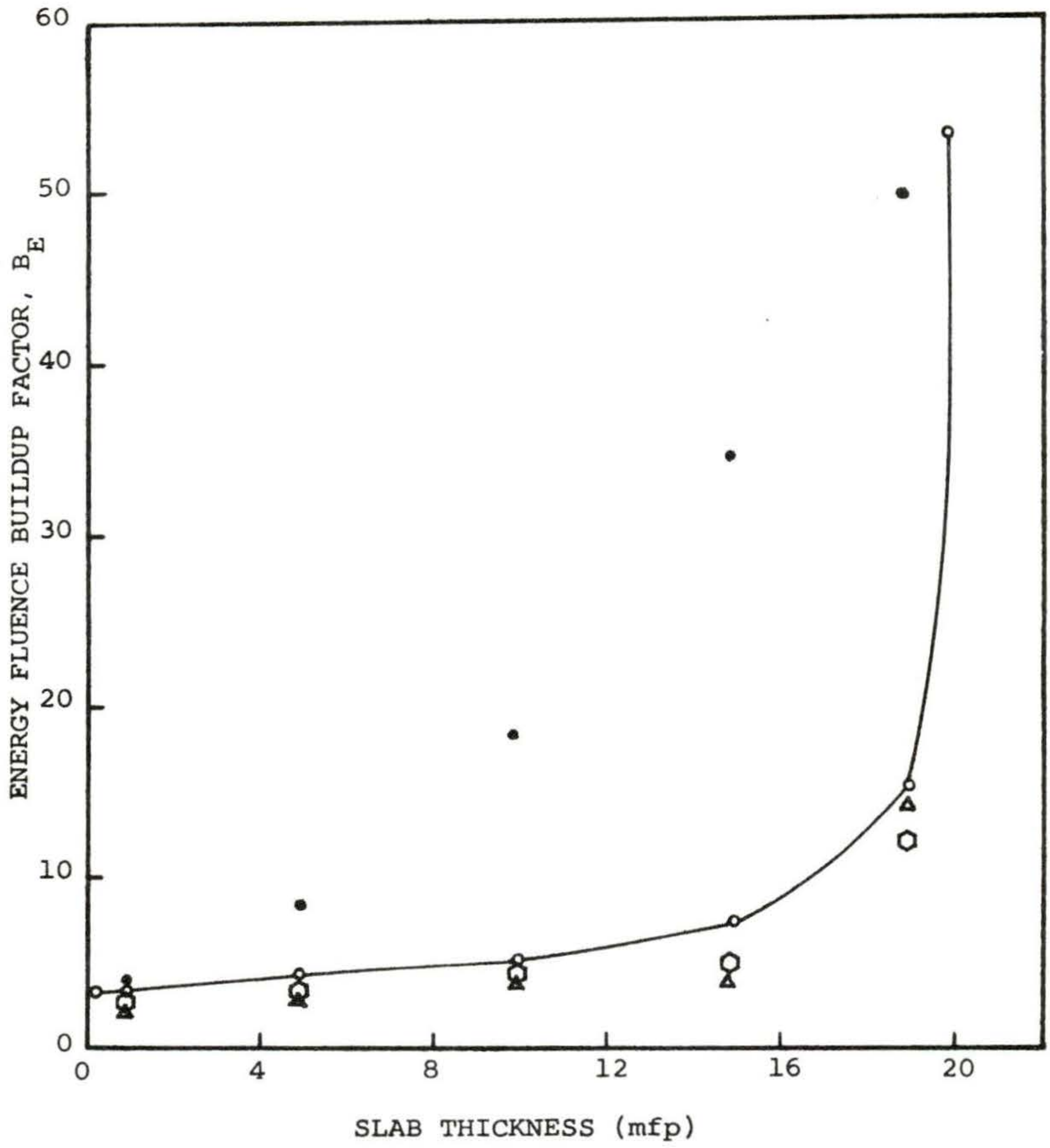
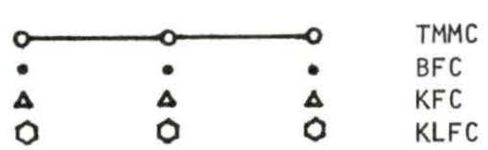



Figure 18. Energy fluence buildup factor of a 20 mfp water-uranium shield for a 1 Mev point isotropic source



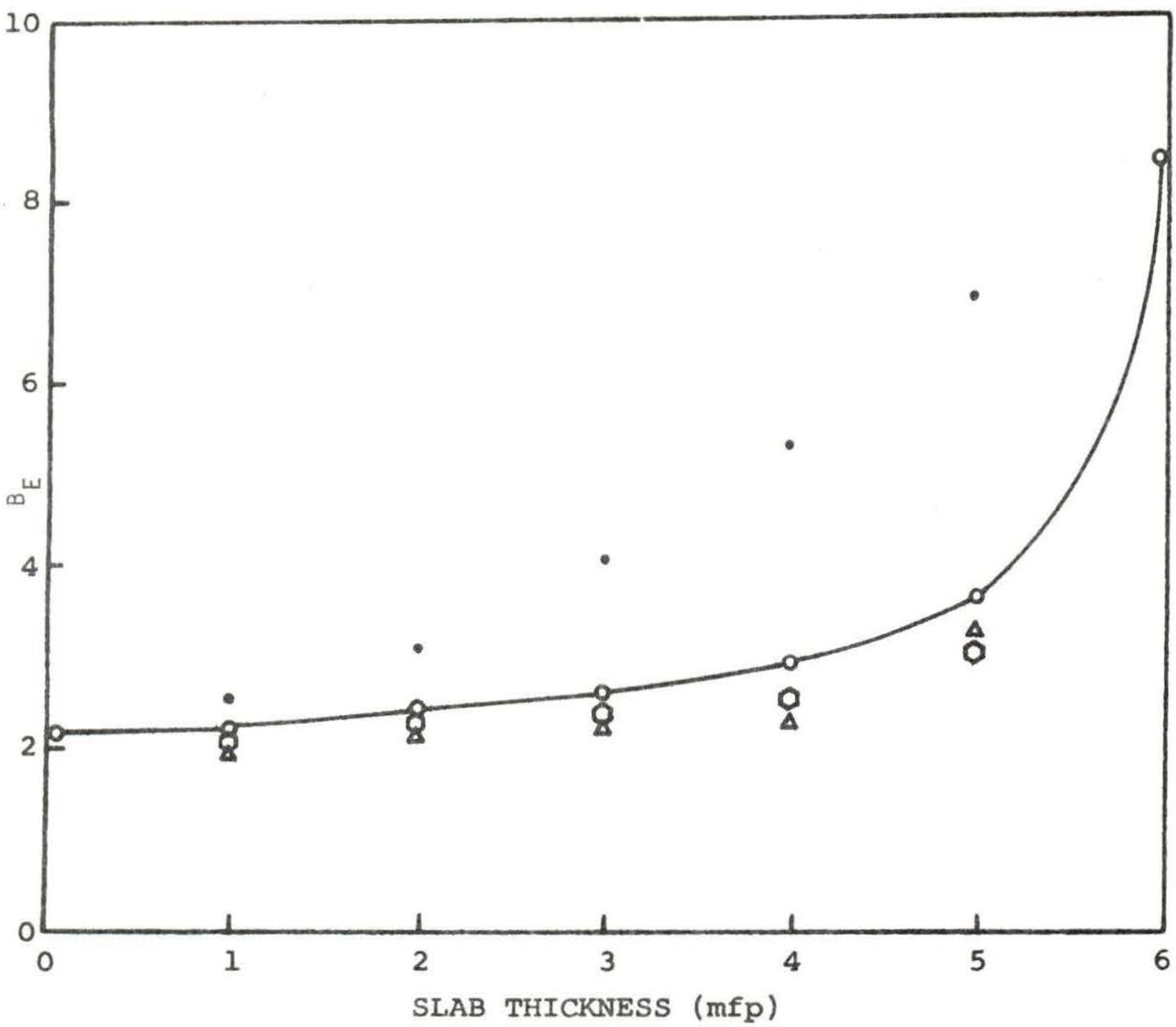
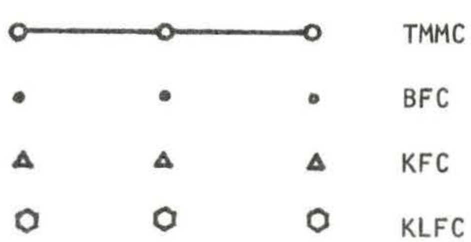



Figure 19. Energy fluence buildup factor of a 6 mfp water-uranium shield for a 1 Mev point isotropic source



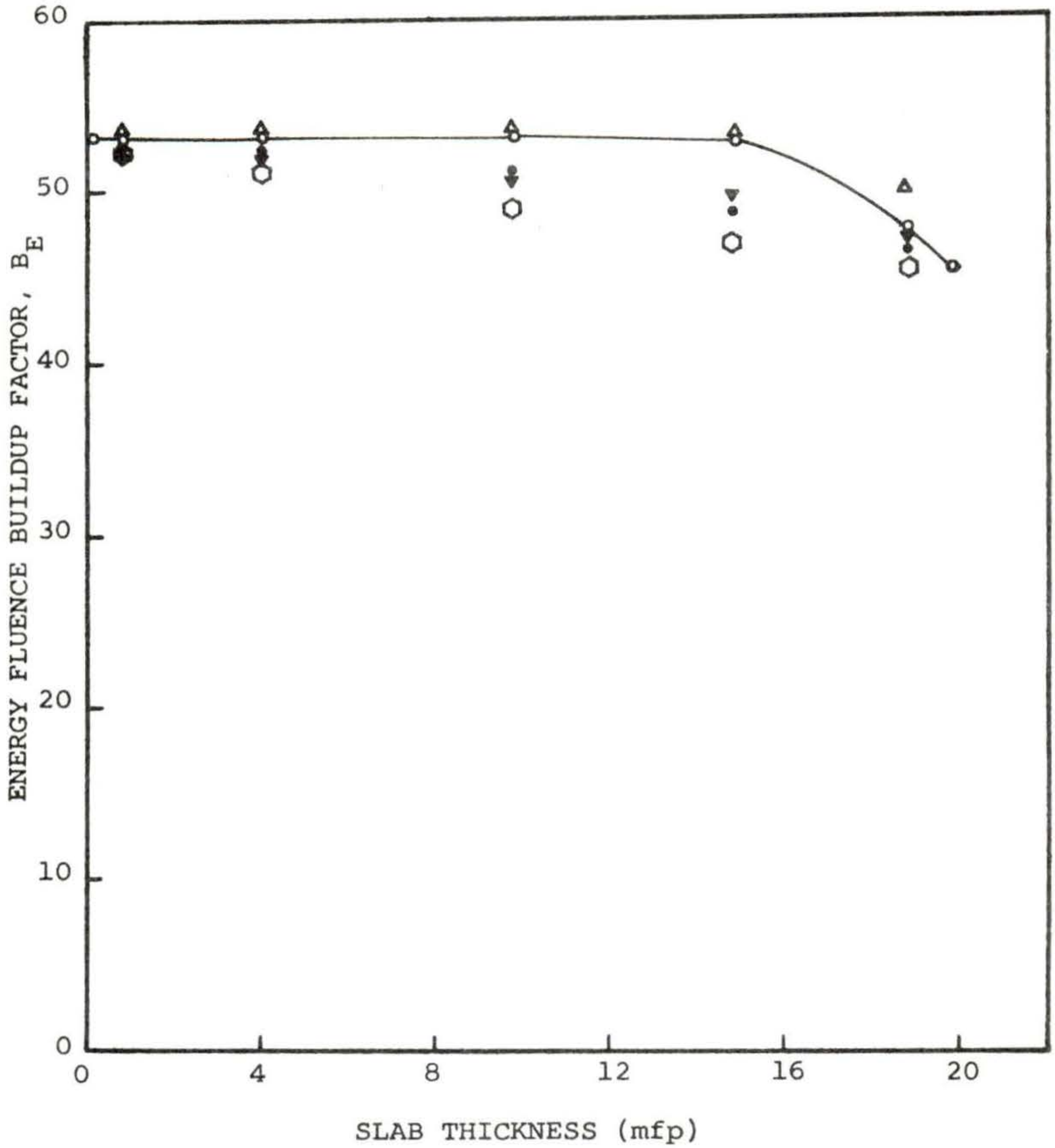
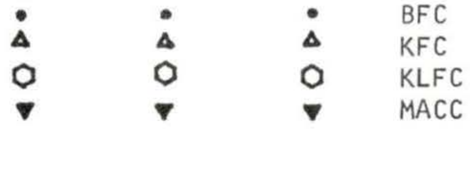



Figure 20. Energy fluence buildup factor of a 20 mfp aluminum-water shield for a 1 Mev point isotropic source



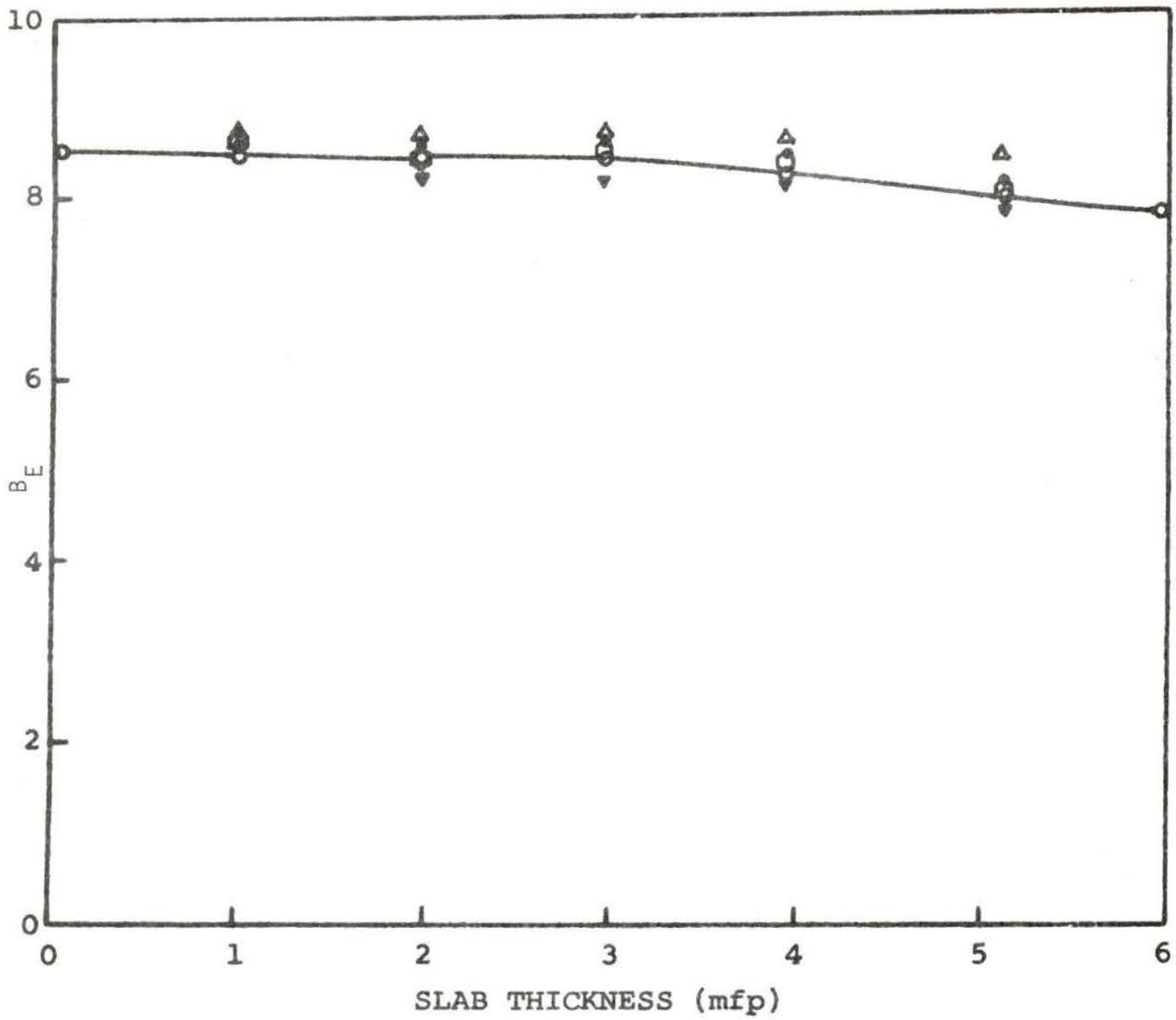
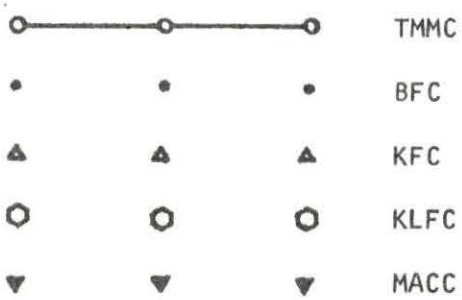



Figure 21. Energy fluence buildup factor of a 6 mfp aluminum-water shield for a 1 Mev point isotropic source



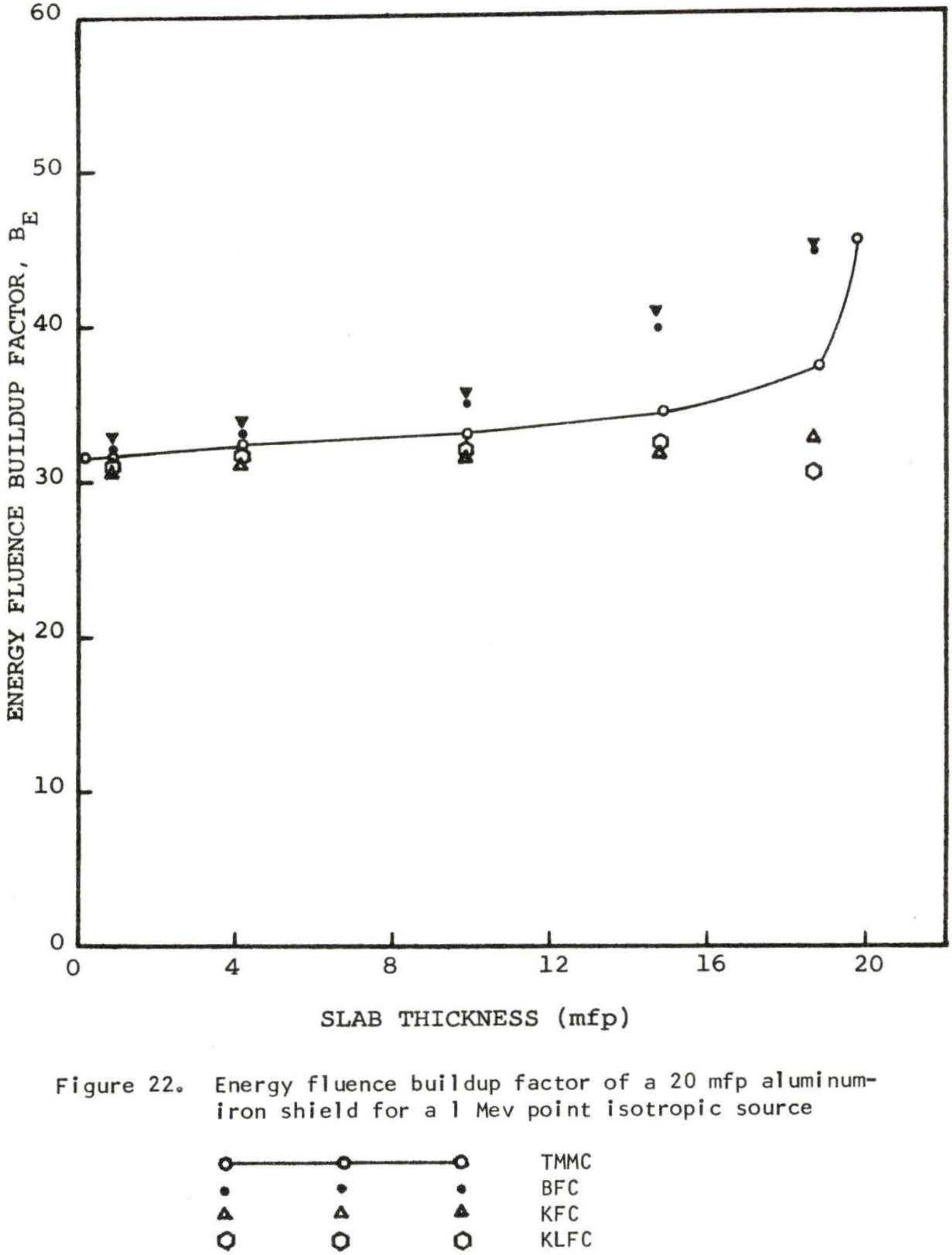



Figure 22. Energy fluence buildup factor of a 20 mfp aluminum-iron shield for a 1 Mev point isotropic source

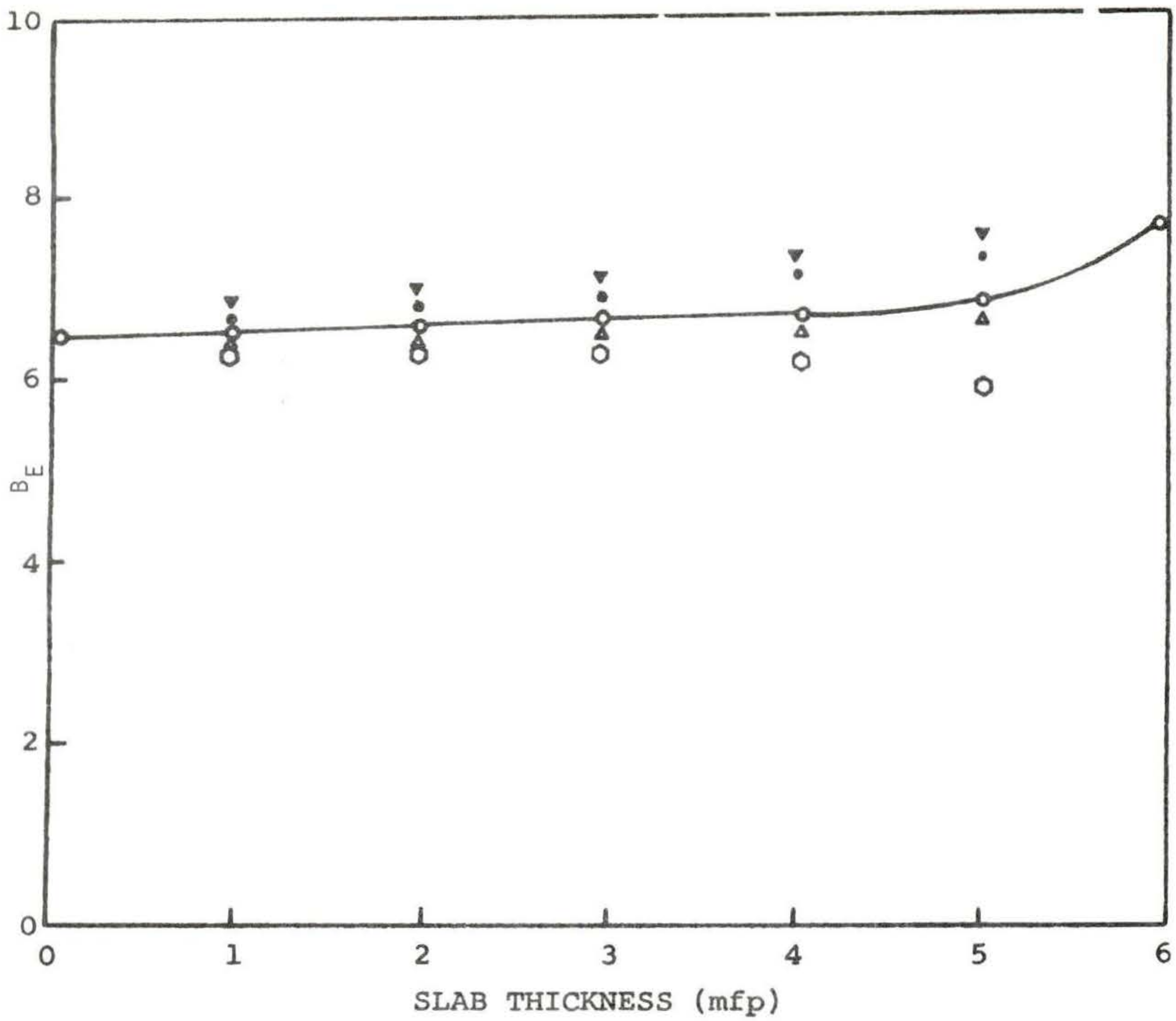
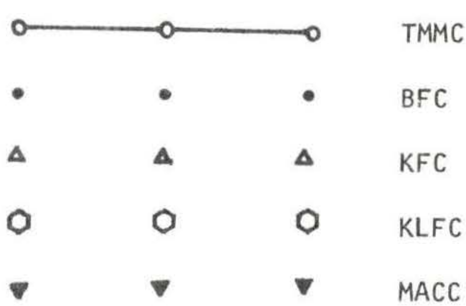



Figure 23. Energy fluence buildup factor for a 6 mfp aluminum-iron shield for a 1 Mev point isotropic source



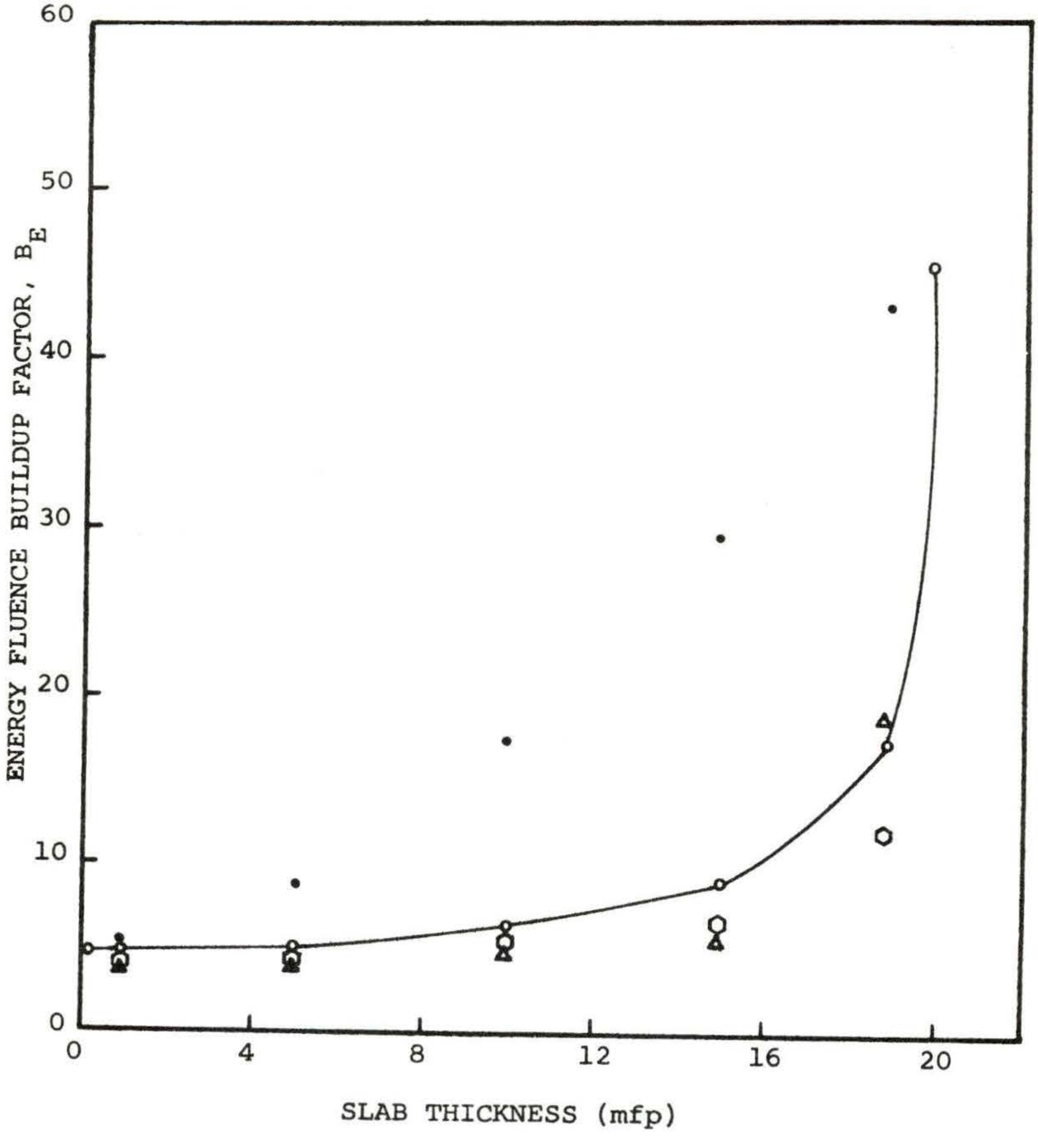
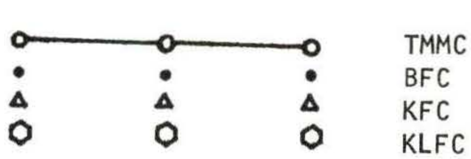



Figure 24. Energy fluence buildup factor of a 20 mfp aluminum-lead shield for a 1 Mev point isotropic source



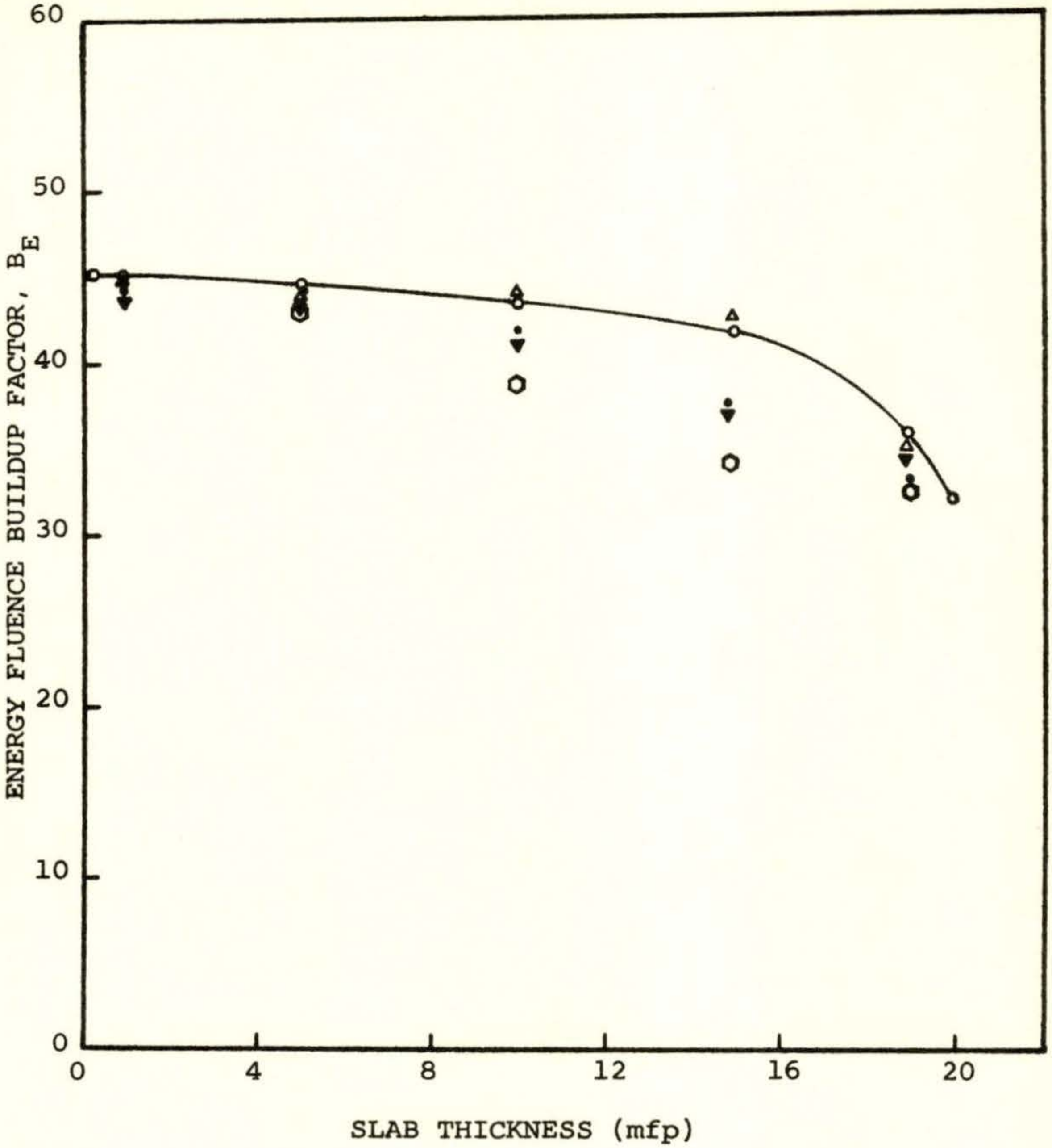
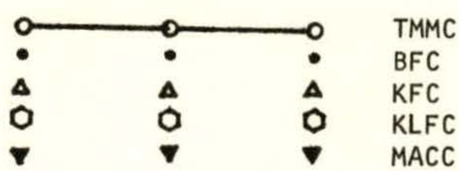



Figure 30. Energy fluence buildup factor of a 20 mfp iron-aluminum shield for a 1 Mev point isotropic source



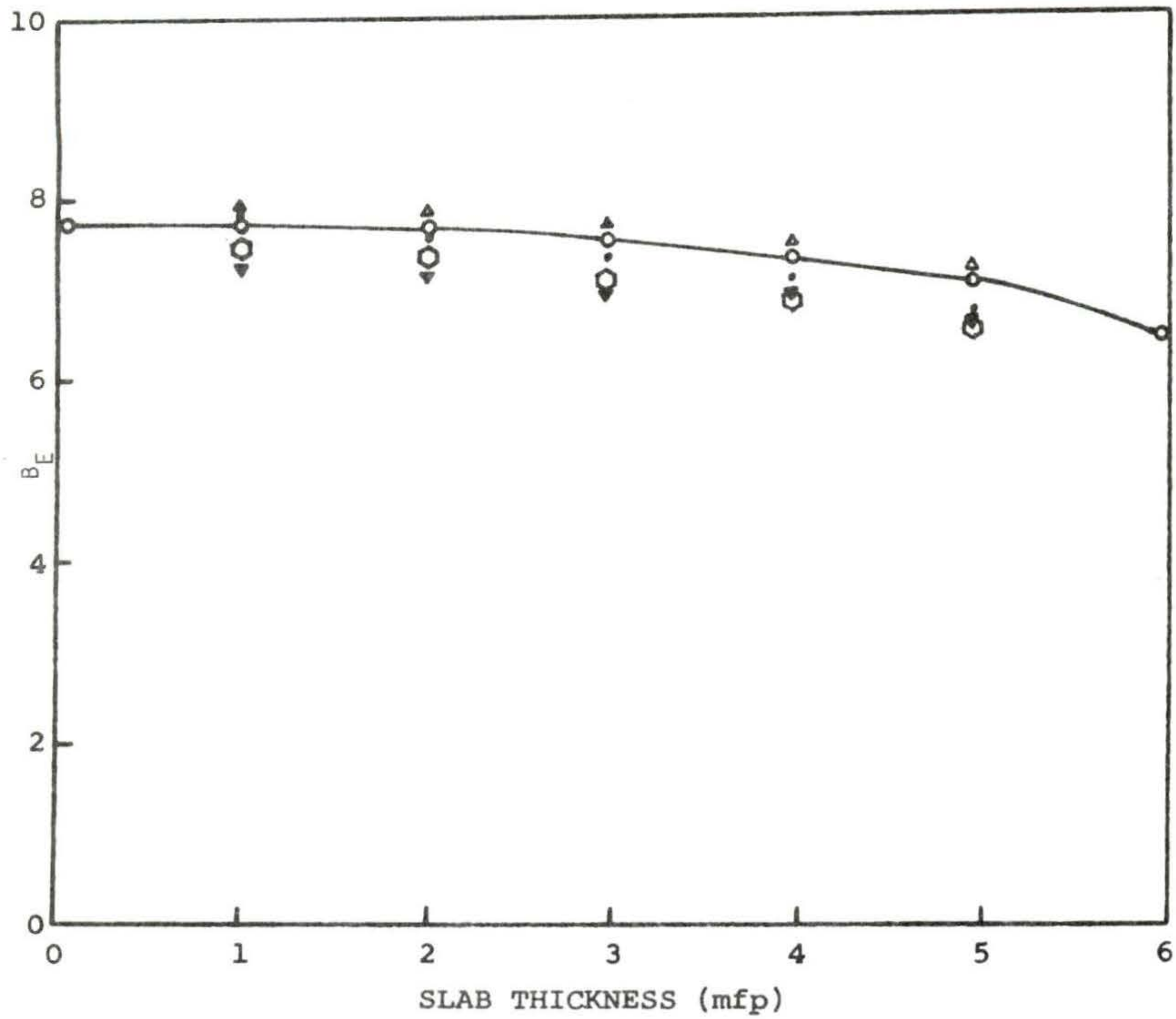
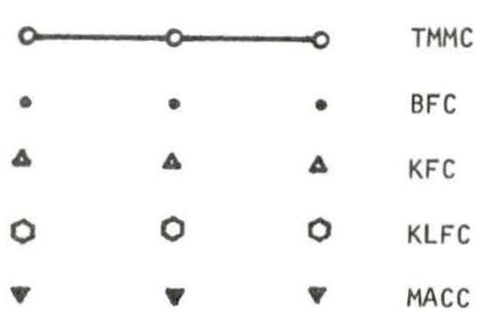



Figure 31. Energy fluence buildup factor of a 6 mfp iron-aluminum shield for a 1 Mev point isotropic source



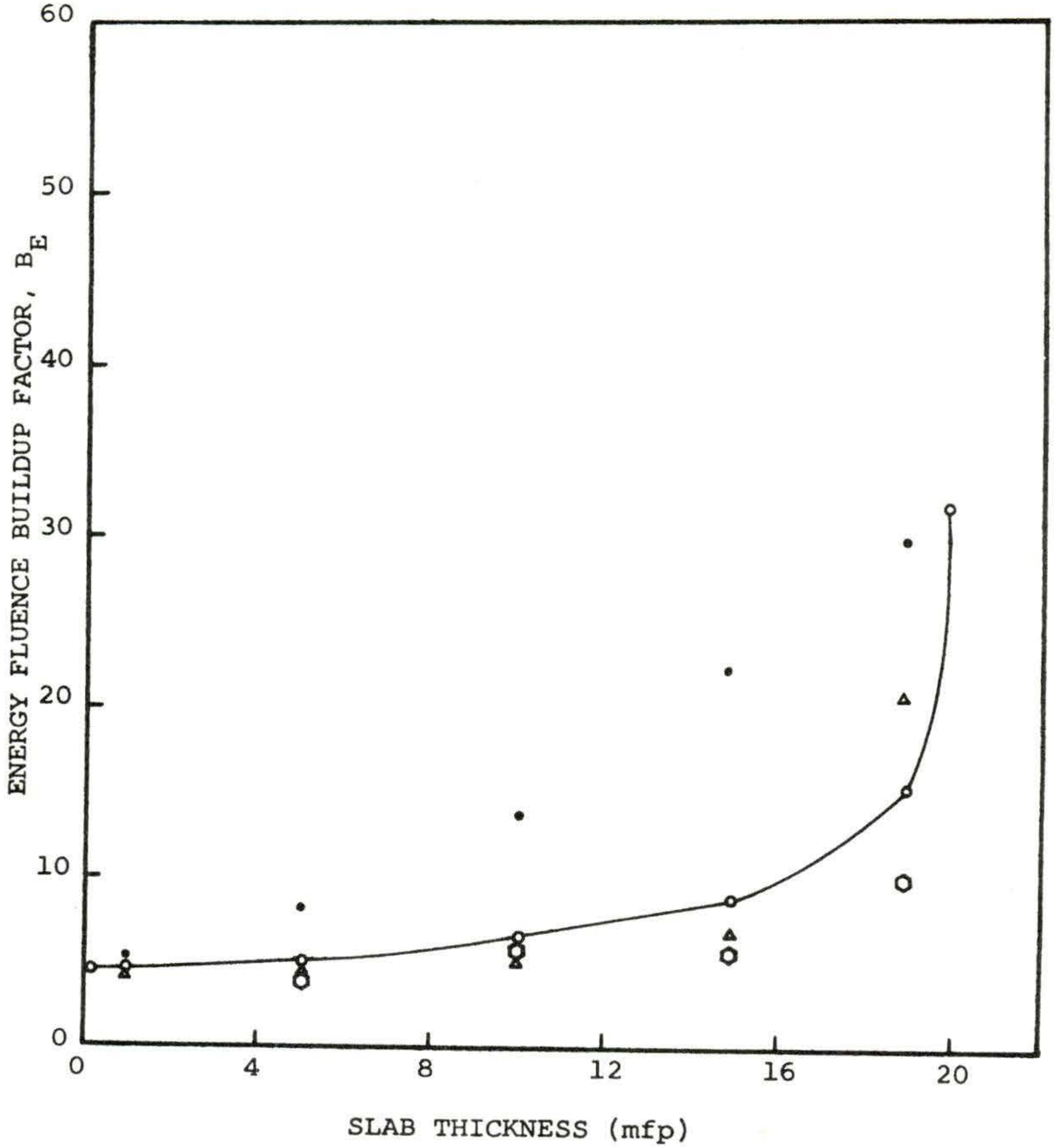



Figure 32. Energy fluence buildup factor of a 20 mfp iron-lead shield for a 1 Mev point isotropic source

TMMC
BFC
KFC
KLFC

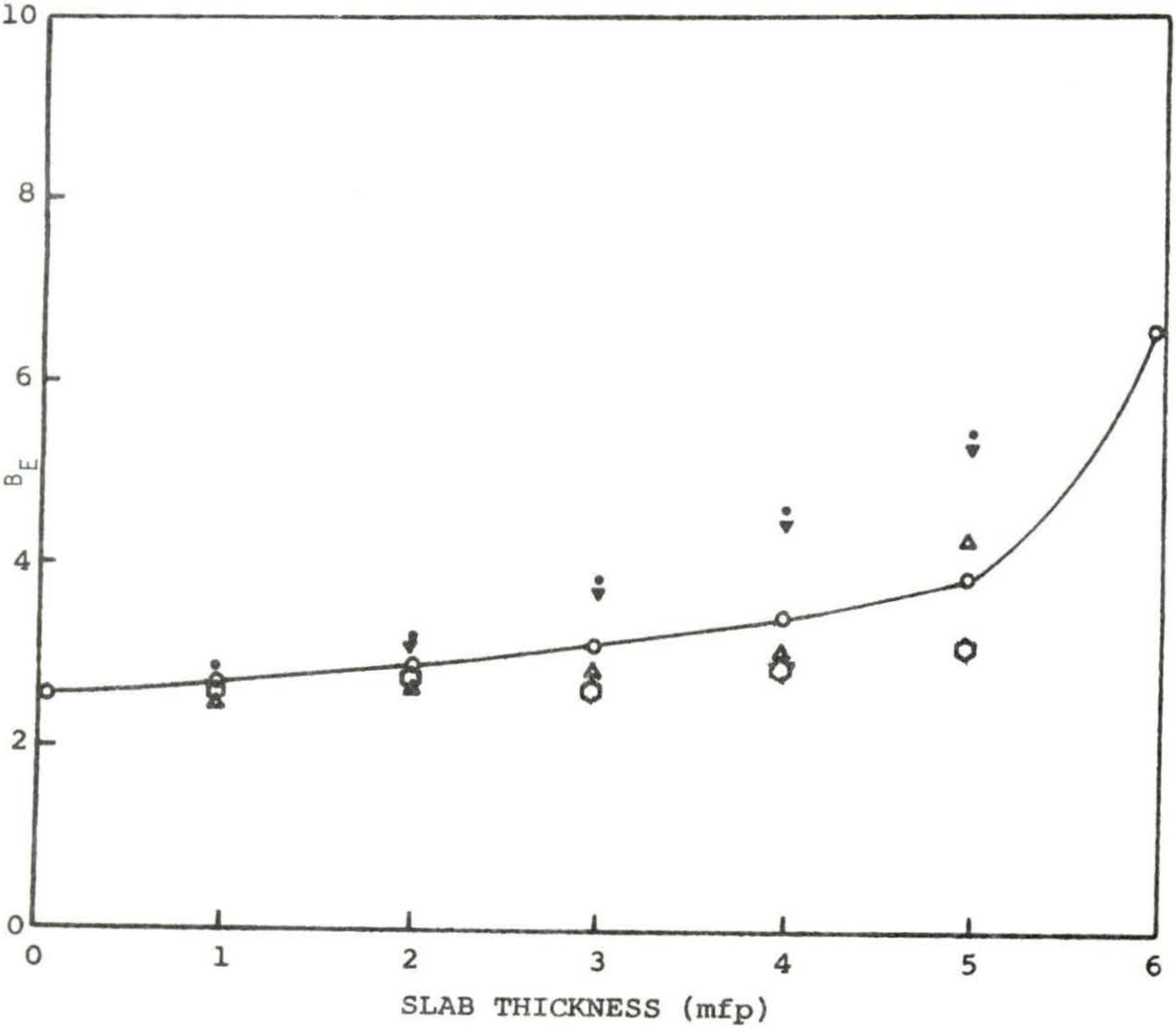



Figure 33. Energy fluence buildup factor of a 6 mfp iron-lead shield for a 1 Mev point isotropic source

o———o———o TMMC

• • • BFC

△ △ △ KFC

⬡ ⬡ ⬡ KLFC

▼ ▼ ▼ MACC

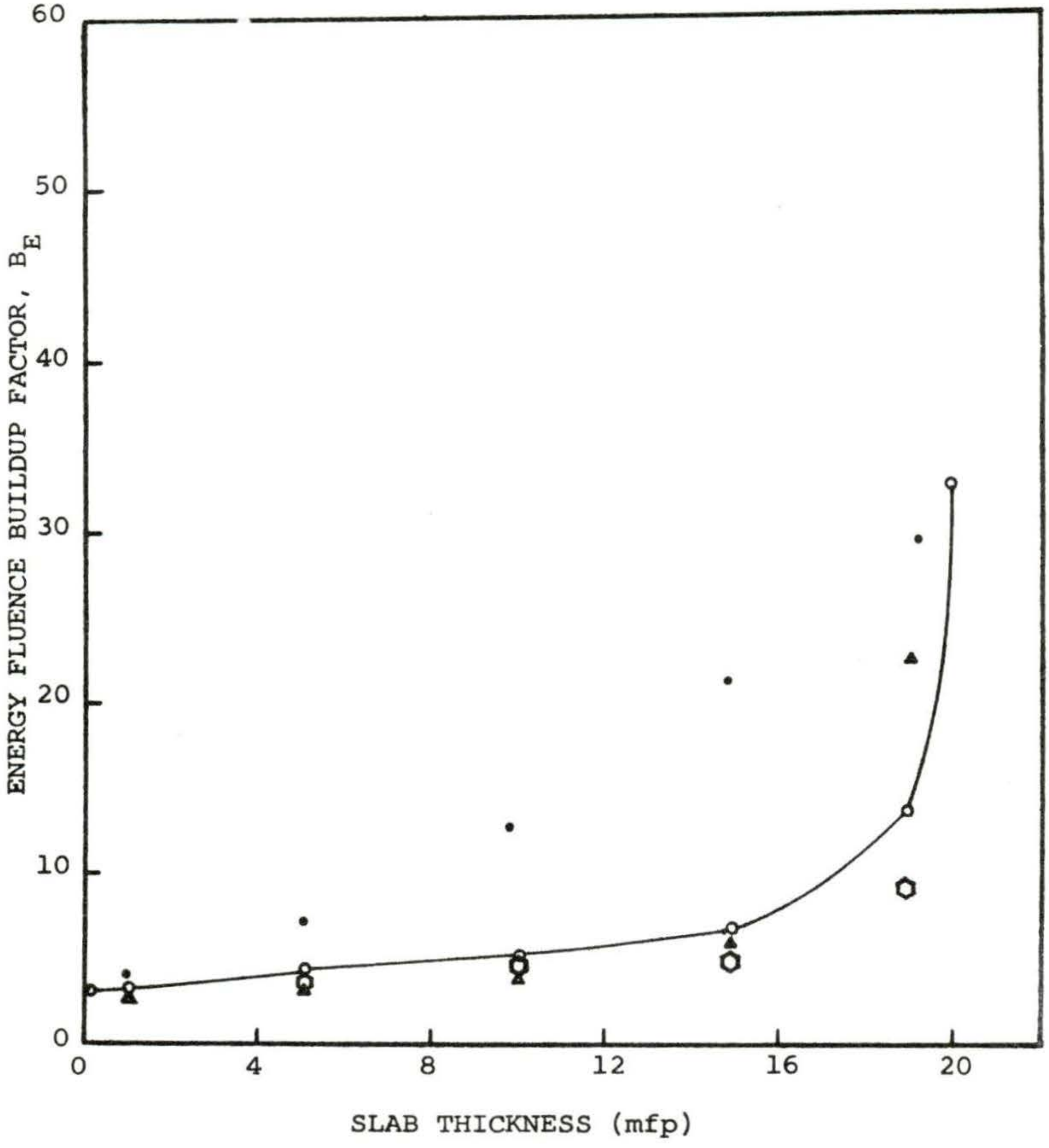



Figure 34. Energy fluence buildup factor of a 20 mfp iron-uranium shield for a 1 Mev point isotropic source

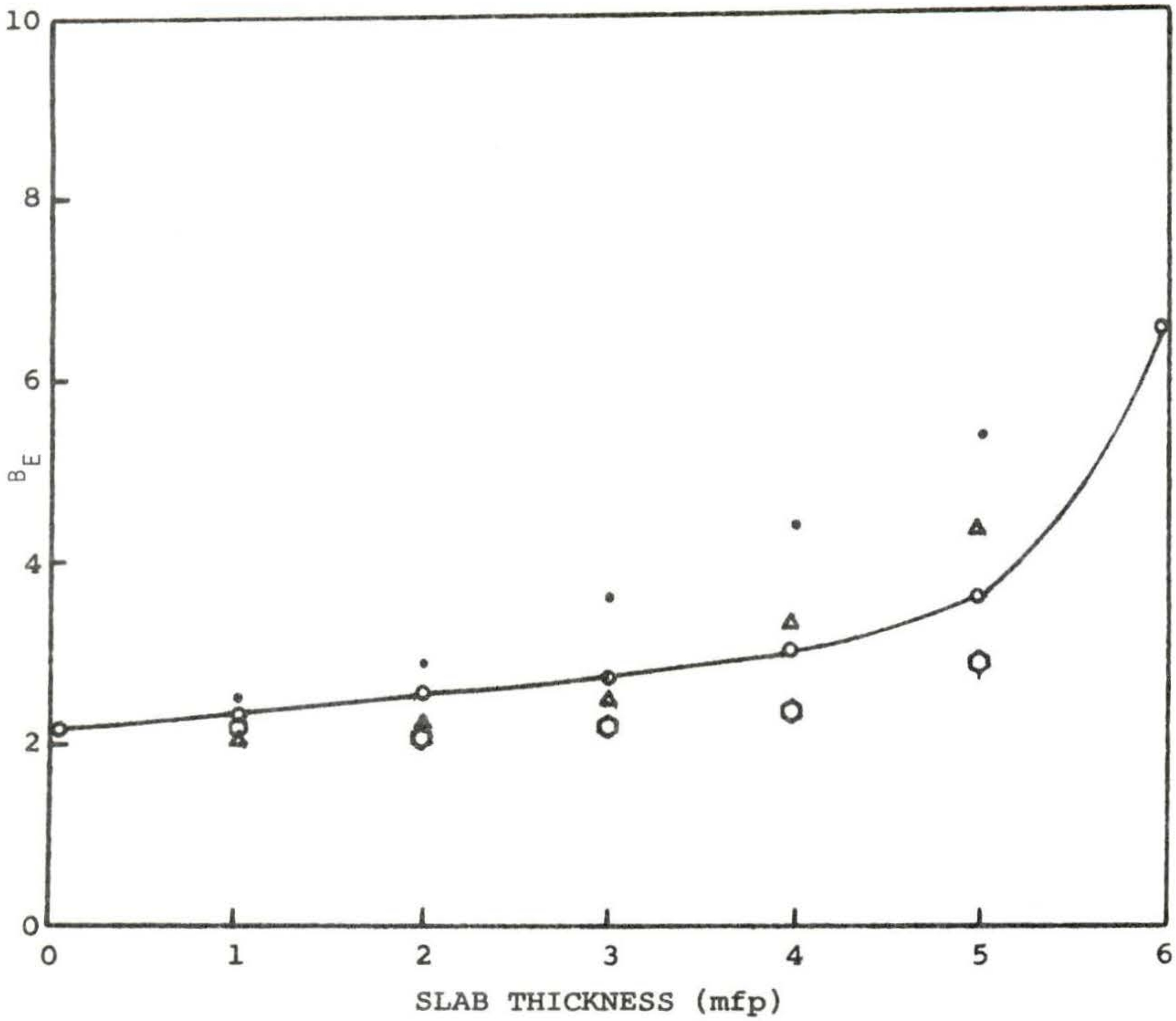
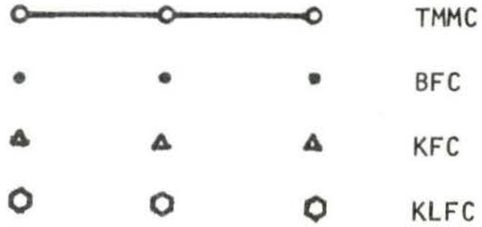



Figure 35. Energy fluence buildup factor of a 6 mfp iron-uranium shield for a 1 Mev point isotropic source



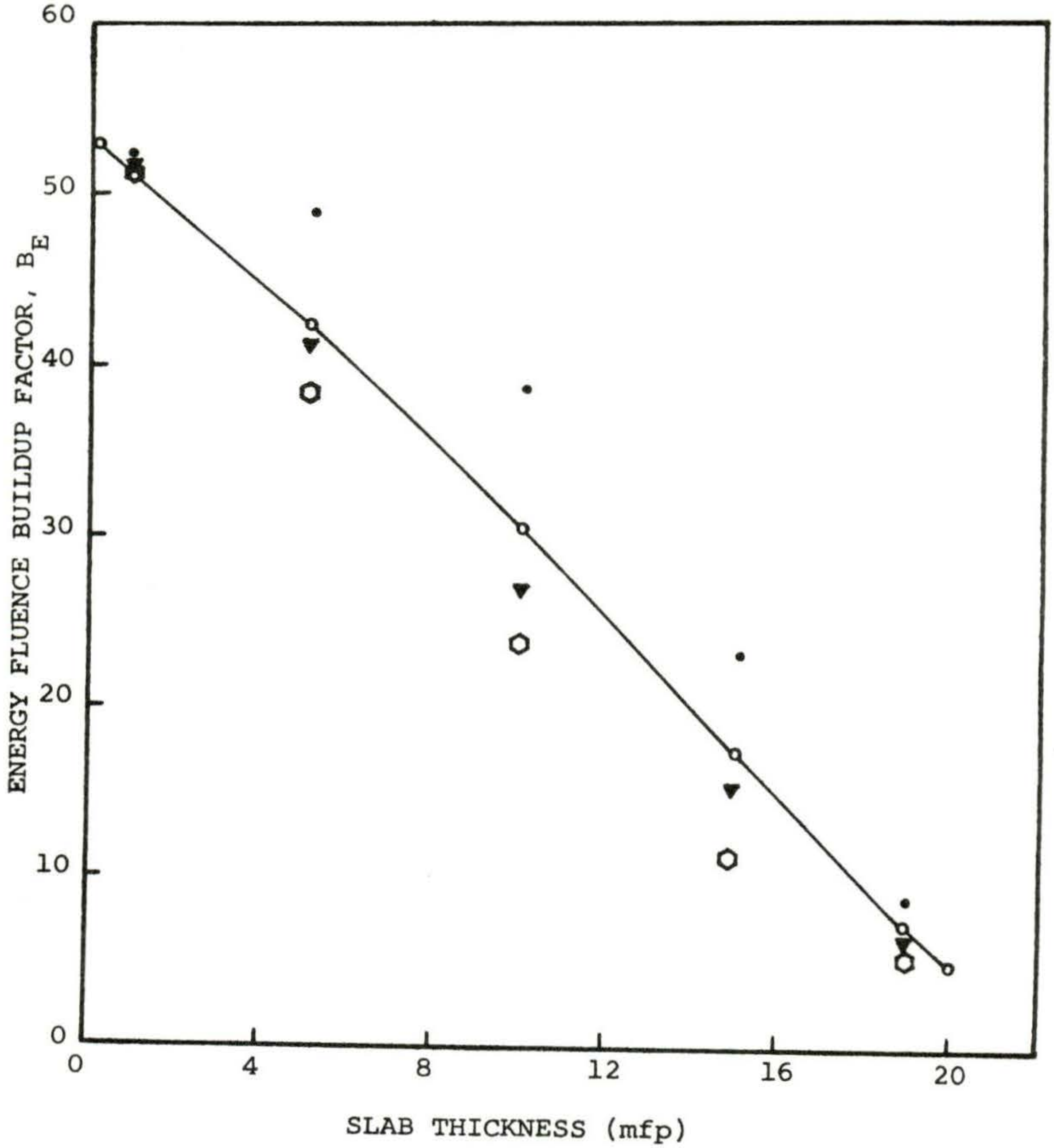
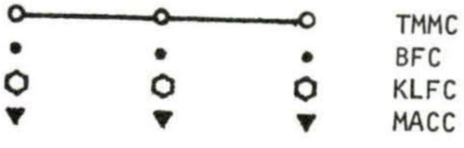



Figure 36. Energy fluence buildup factor of a 20 mfp lead-water shield for a 1 Mev point isotropic source



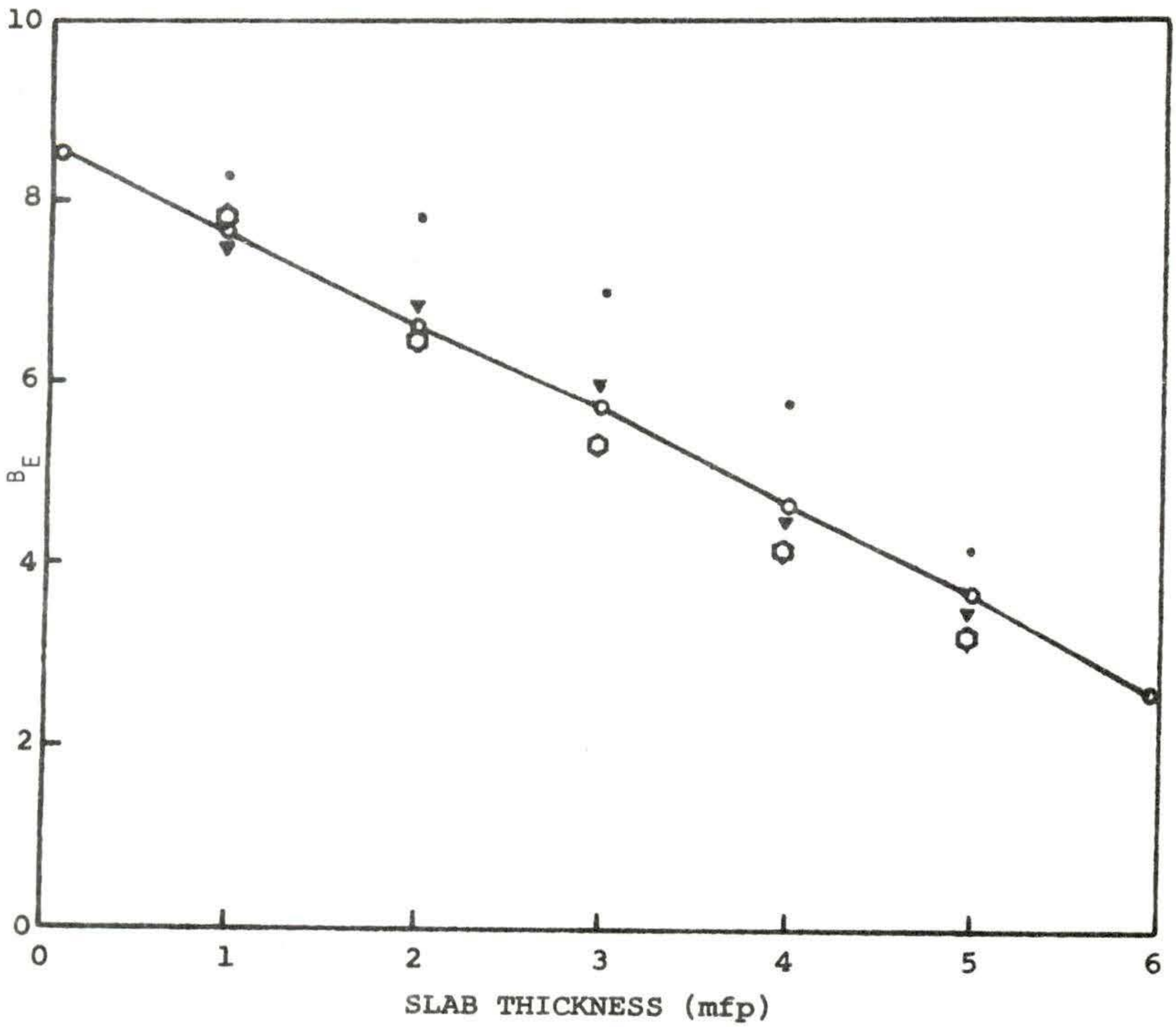
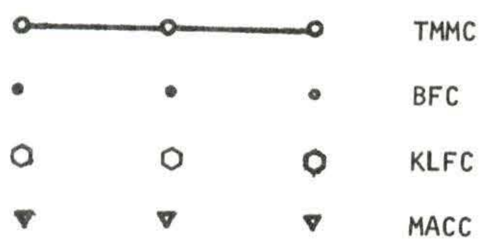



Figure 37. Energy fluence buildup factor of a 6 mfp lead-water shield for a 1 Mev point isotropic source



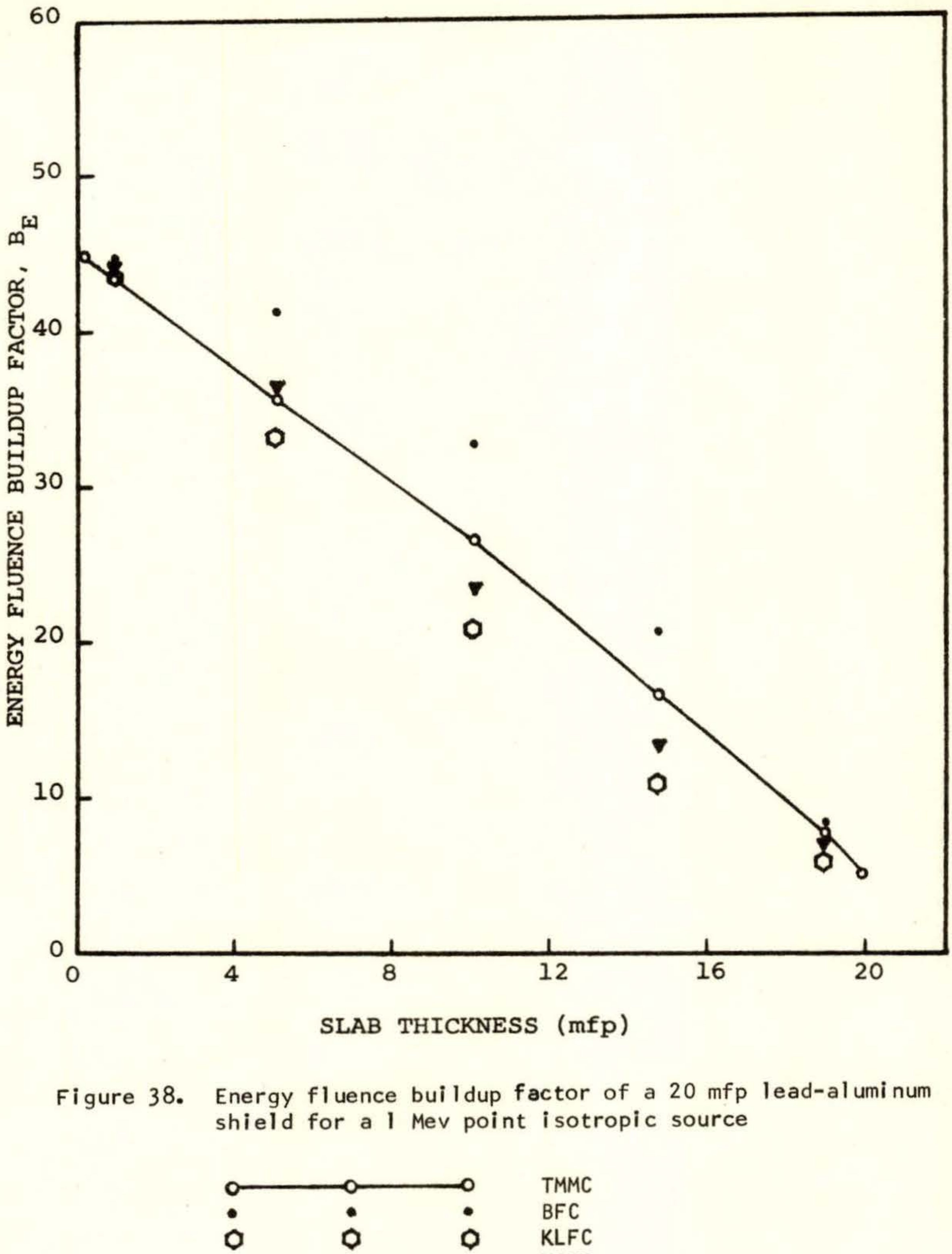



Figure 38. Energy fluence buildup factor of a 20 mfp lead-aluminum shield for a 1 Mev point isotropic source

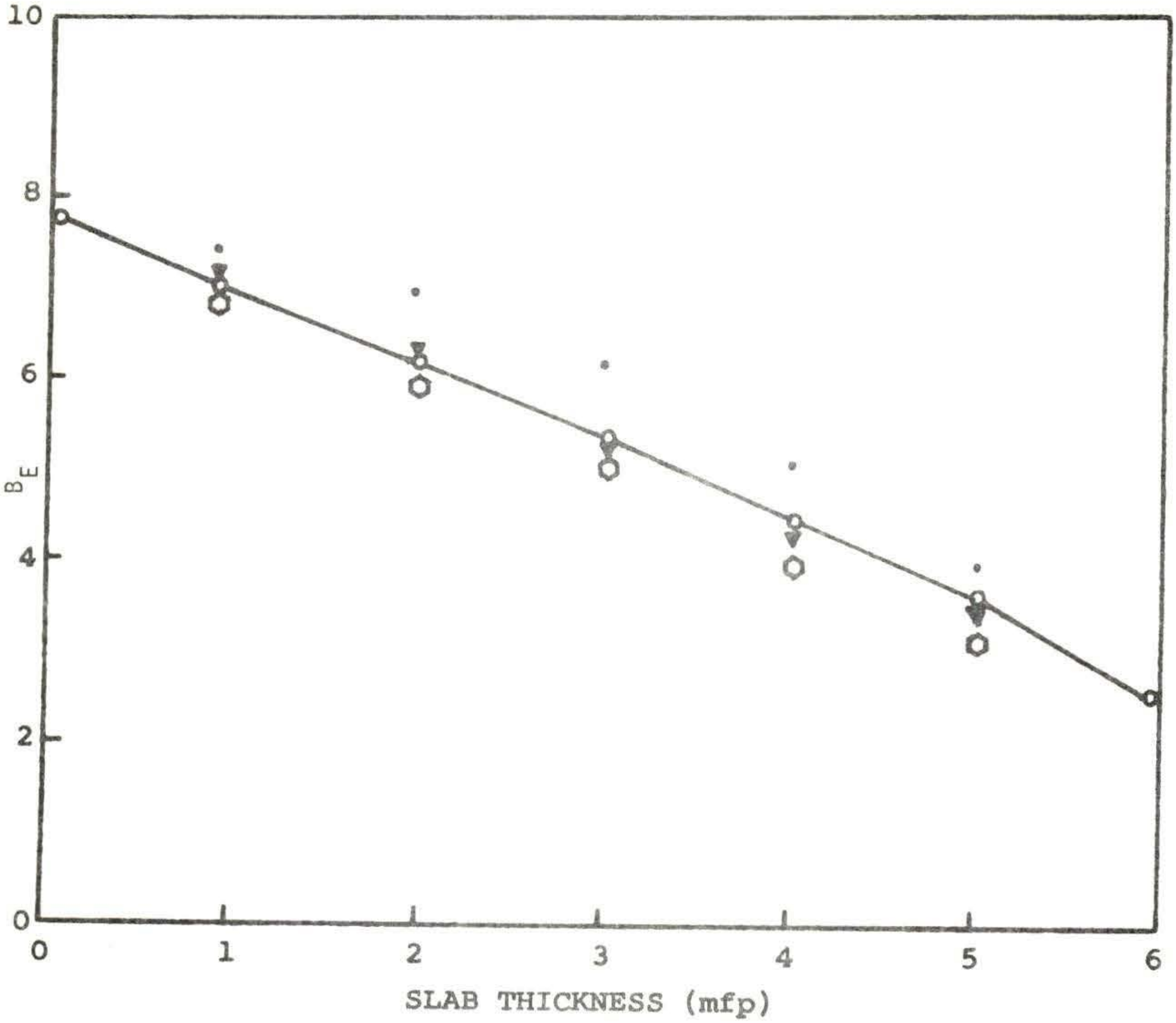
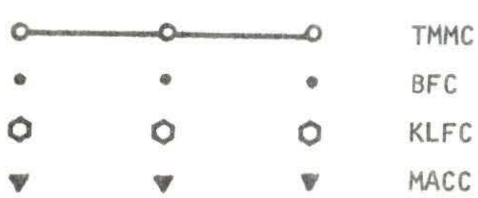



Figure 39. Energy fluence buildup factor of a 6 mfp lead-aluminum shield for a 1 Mev point isotropic source



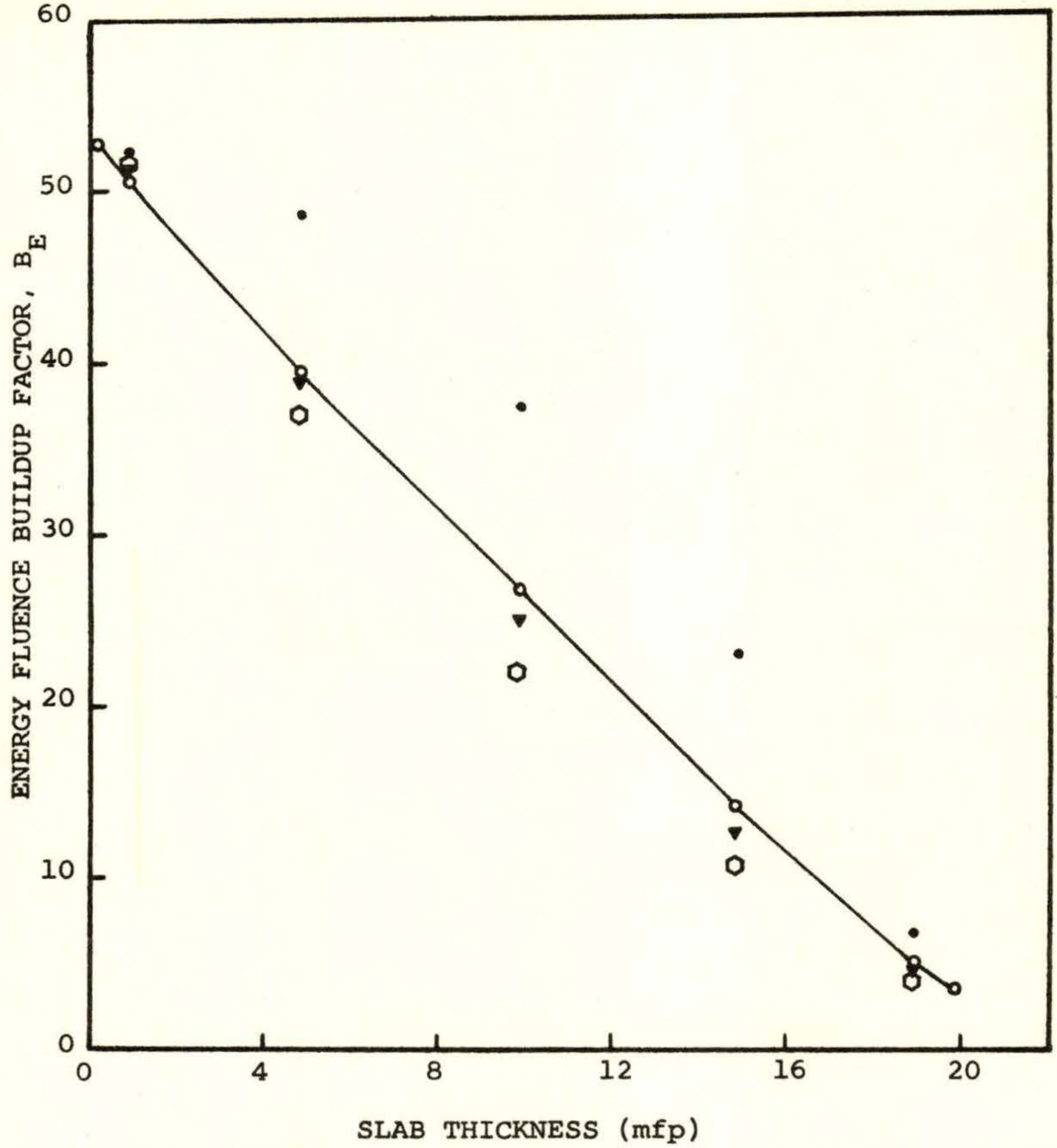
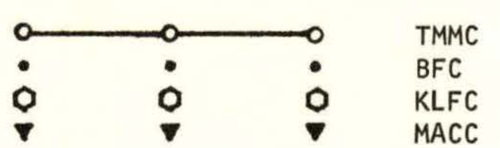



Figure 42. Energy fluence buildup factor of a 20 mfp uranium-water shield for a 1 Mev point isotropic source



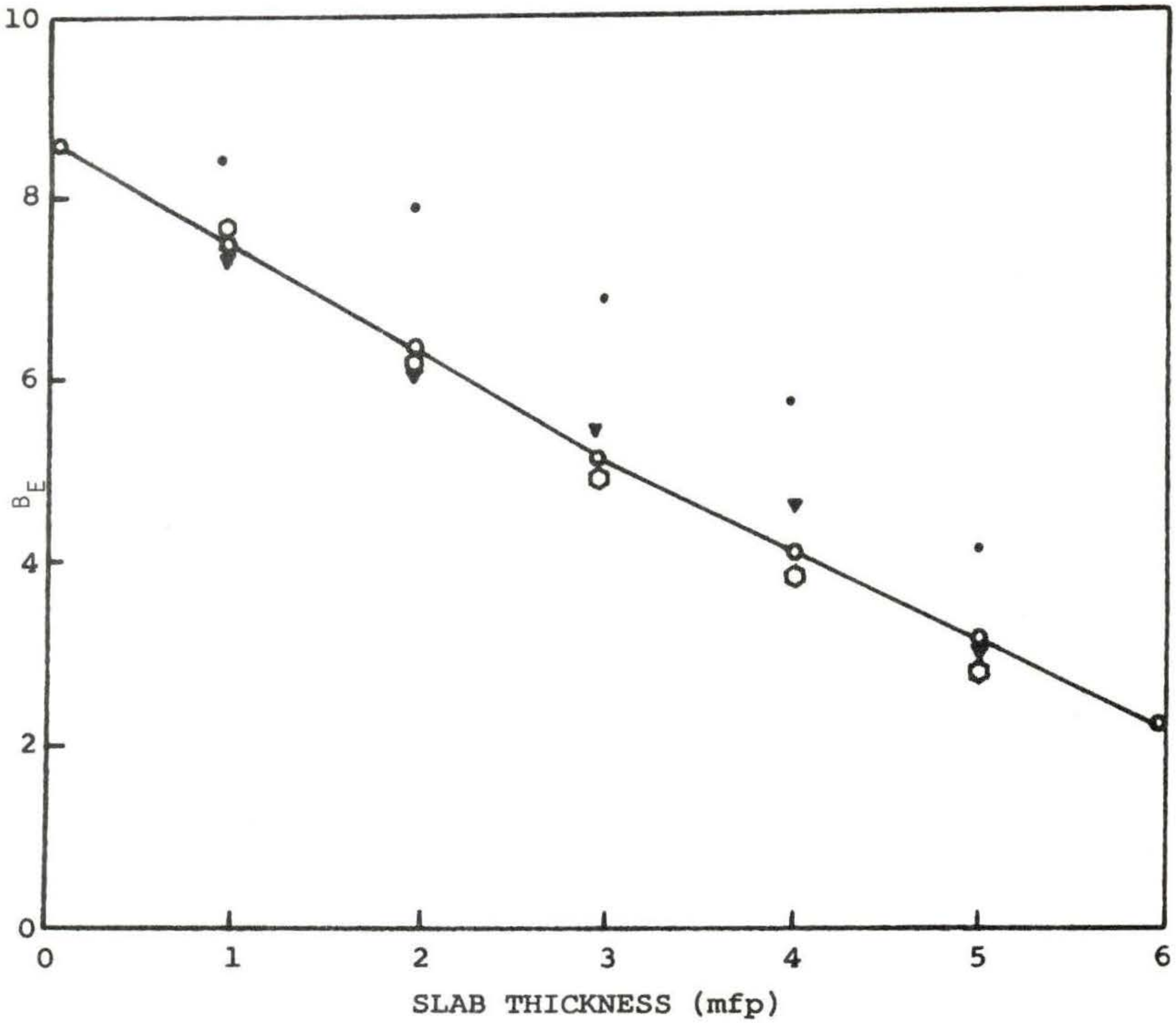
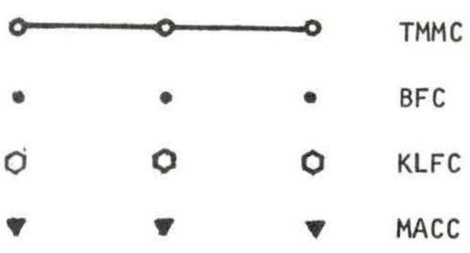



Figure 43. Energy fluence buildup factor of a 6 mfp uranium-water shield for a 1 Mev point isotropic source



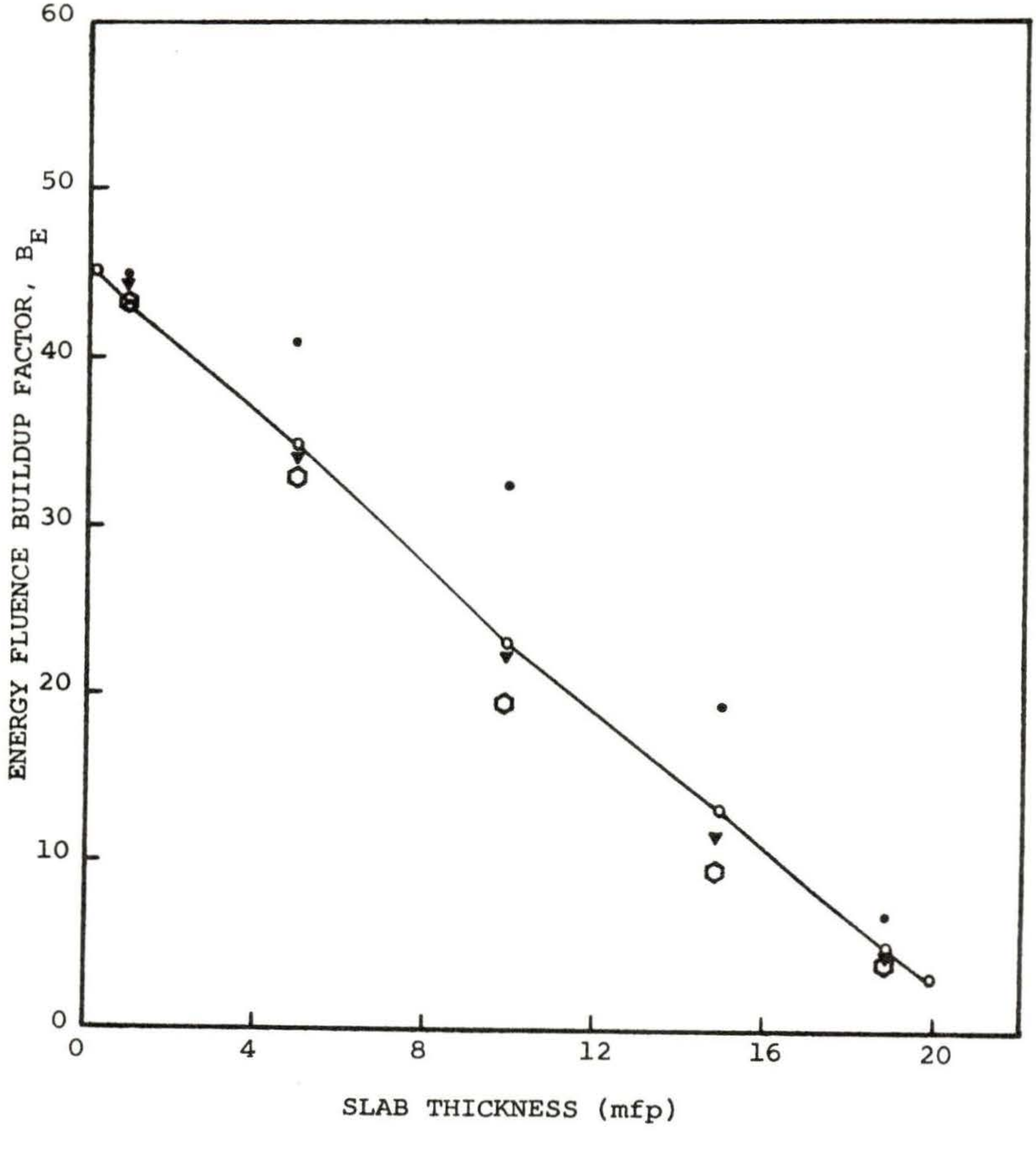
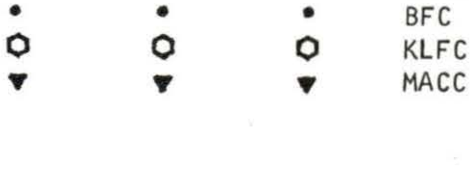



Figure 44. Energy fluence buildup factor of a 20 mfp uranium-aluminum shield for a 1 Mev point isotropic source



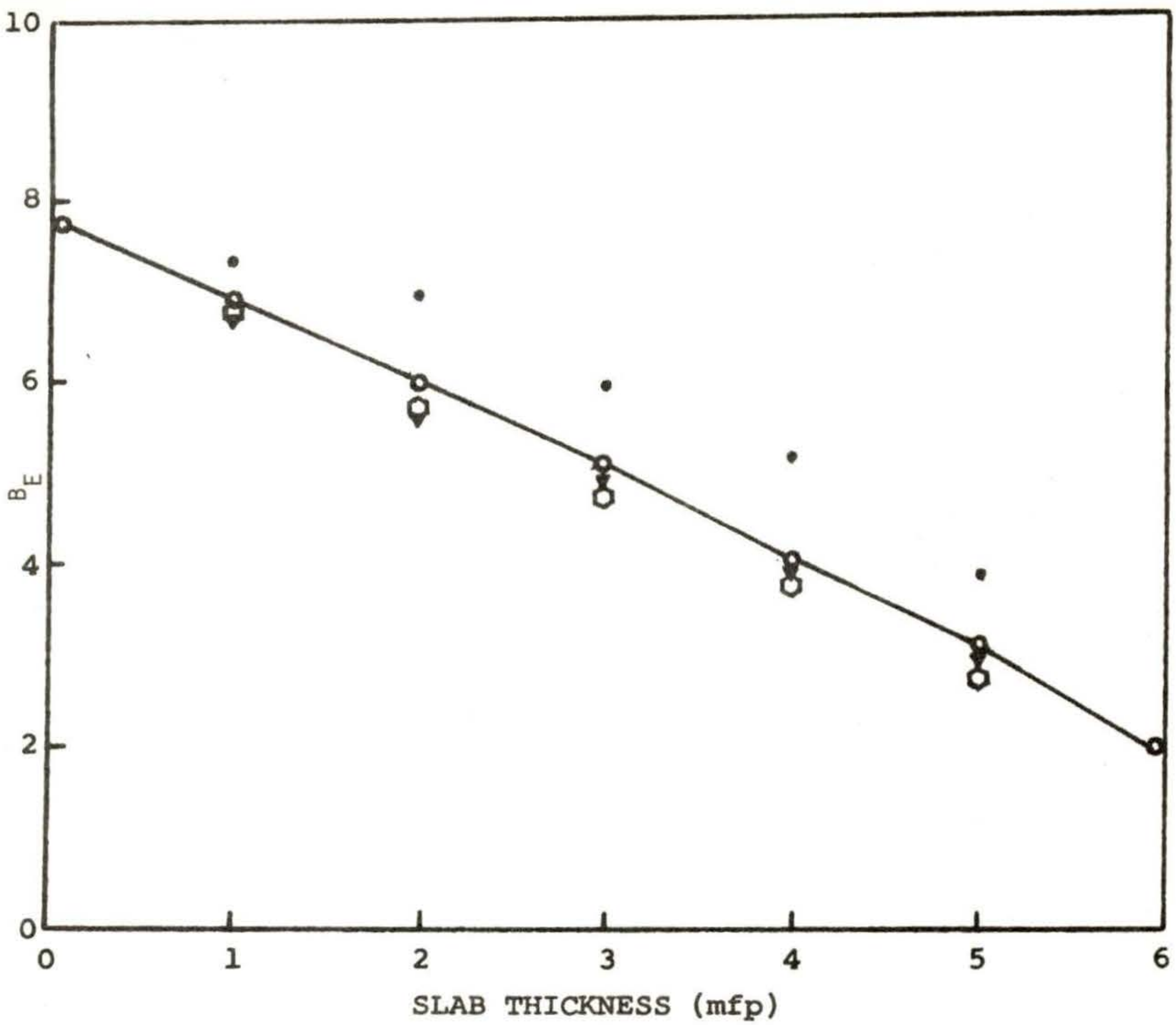
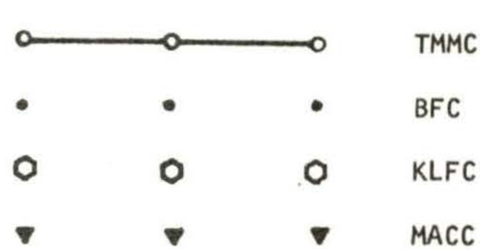



Figure 45. Energy fluence buildup factor of a 6 mfp uranium-aluminum shield for a 1 Mev point isotropic source



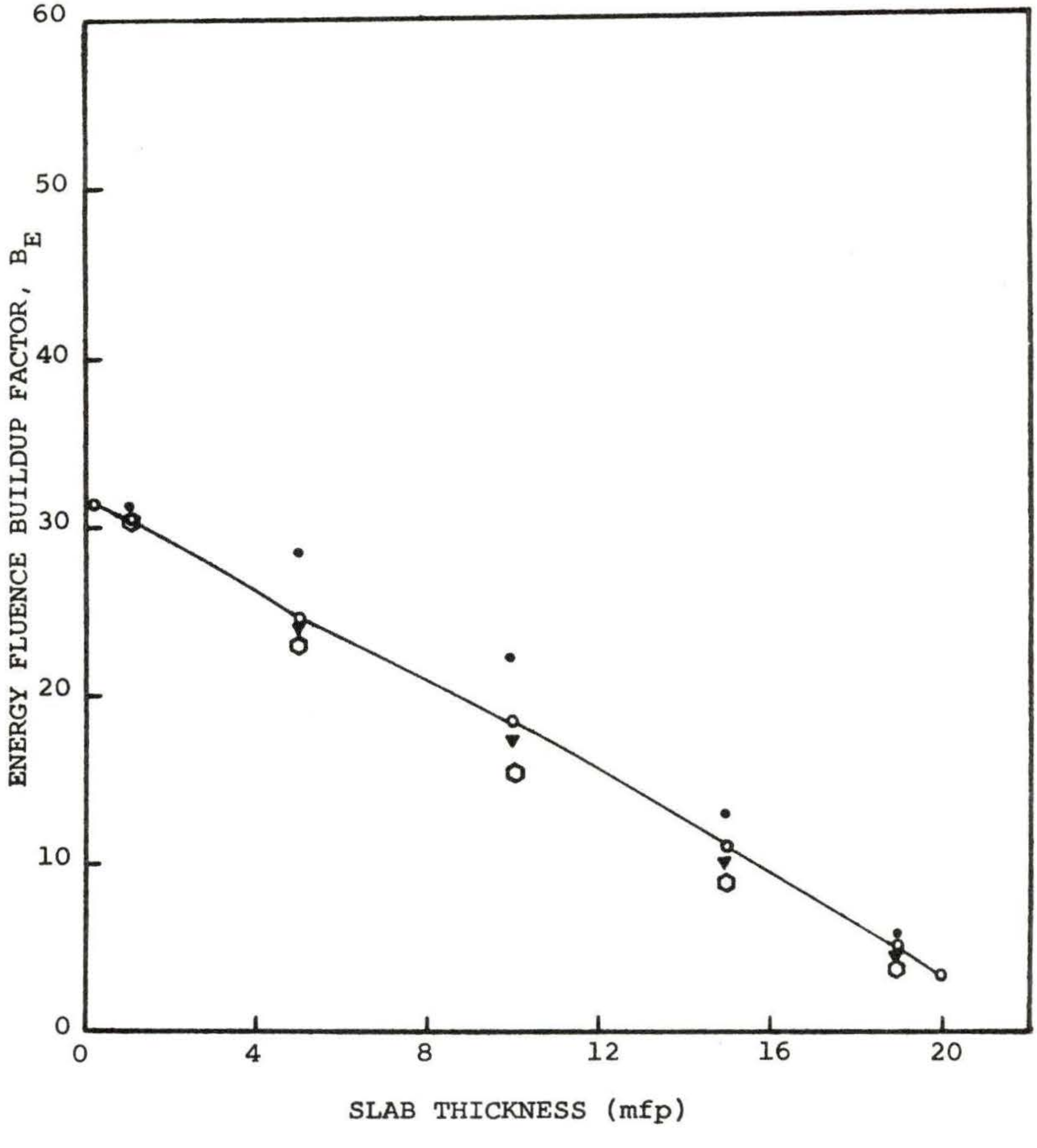
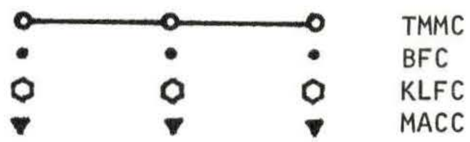



Figure 46. Energy fluence buildup factor of a 20 mfp uranium-iron shield for a 1 Mev point isotropic source



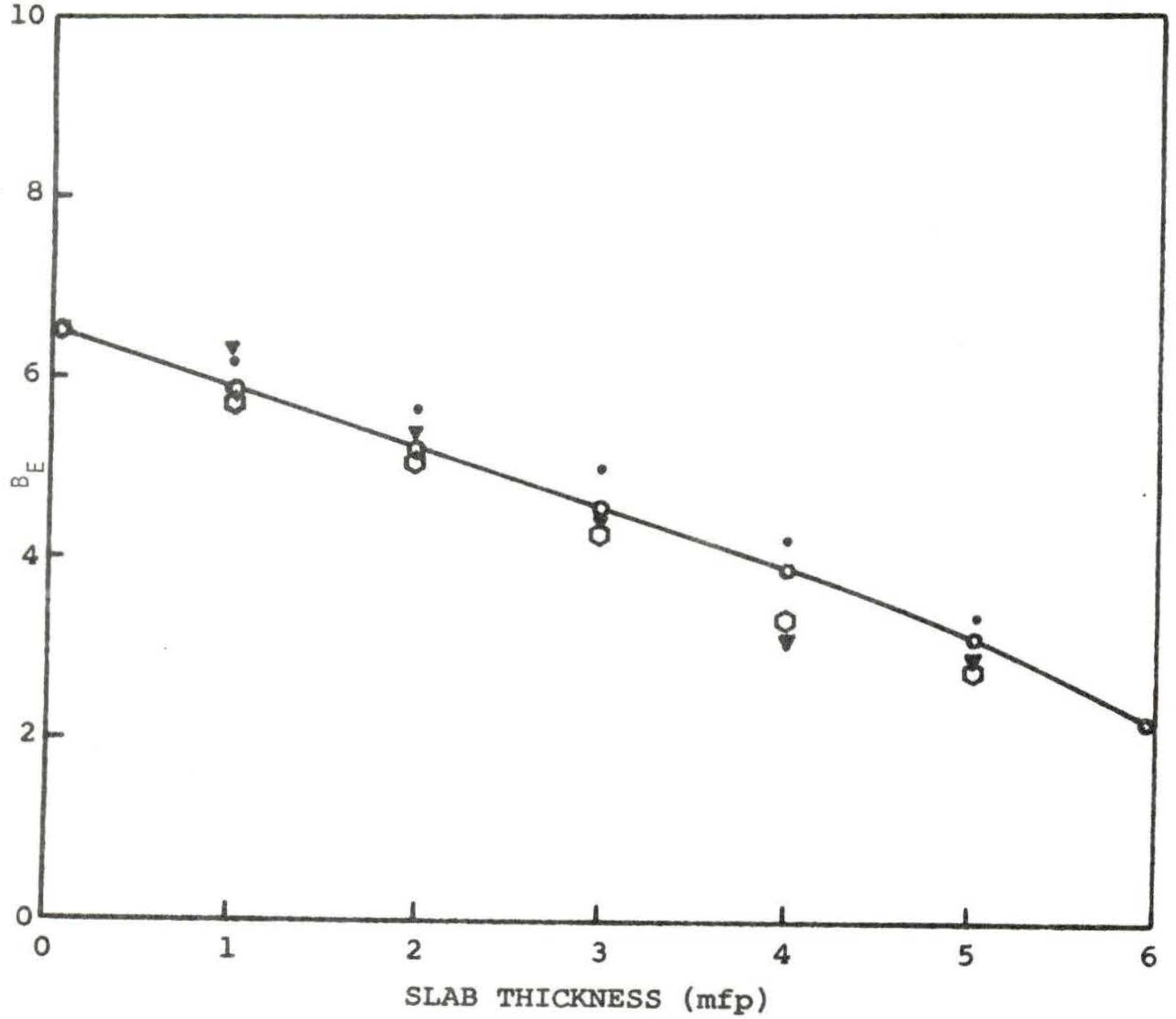



Figure 47. Energy fluence buildup factor of a 6 mfp uranium-iron shield for a 1 Mev point isotropic source

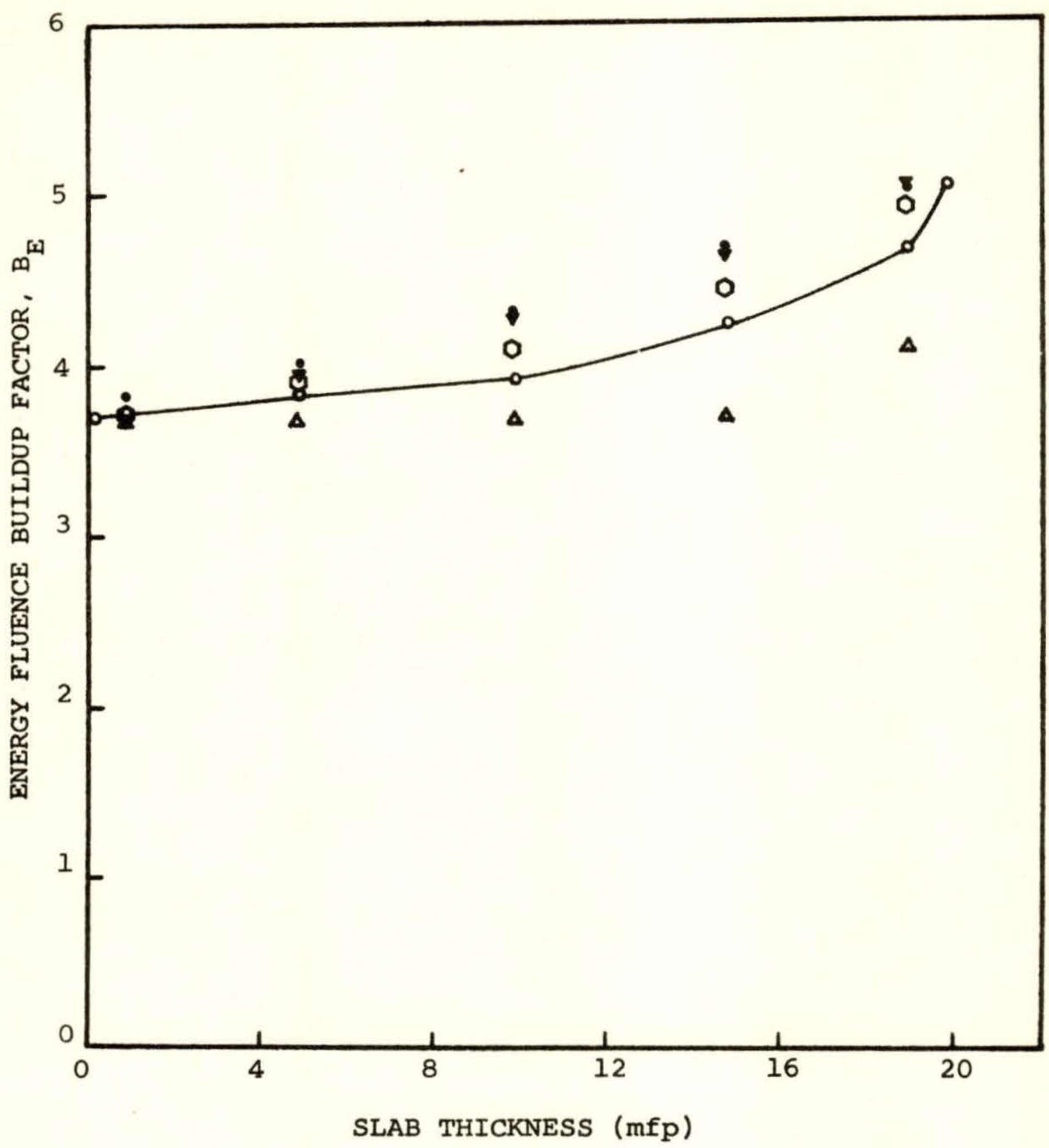



Figure 48. Energy fluence buildup factor of a 20 mfp lead-uranium shield for a 1 Mev point isotropic source

TMMC
BFC
KFC
KLFC
MACC

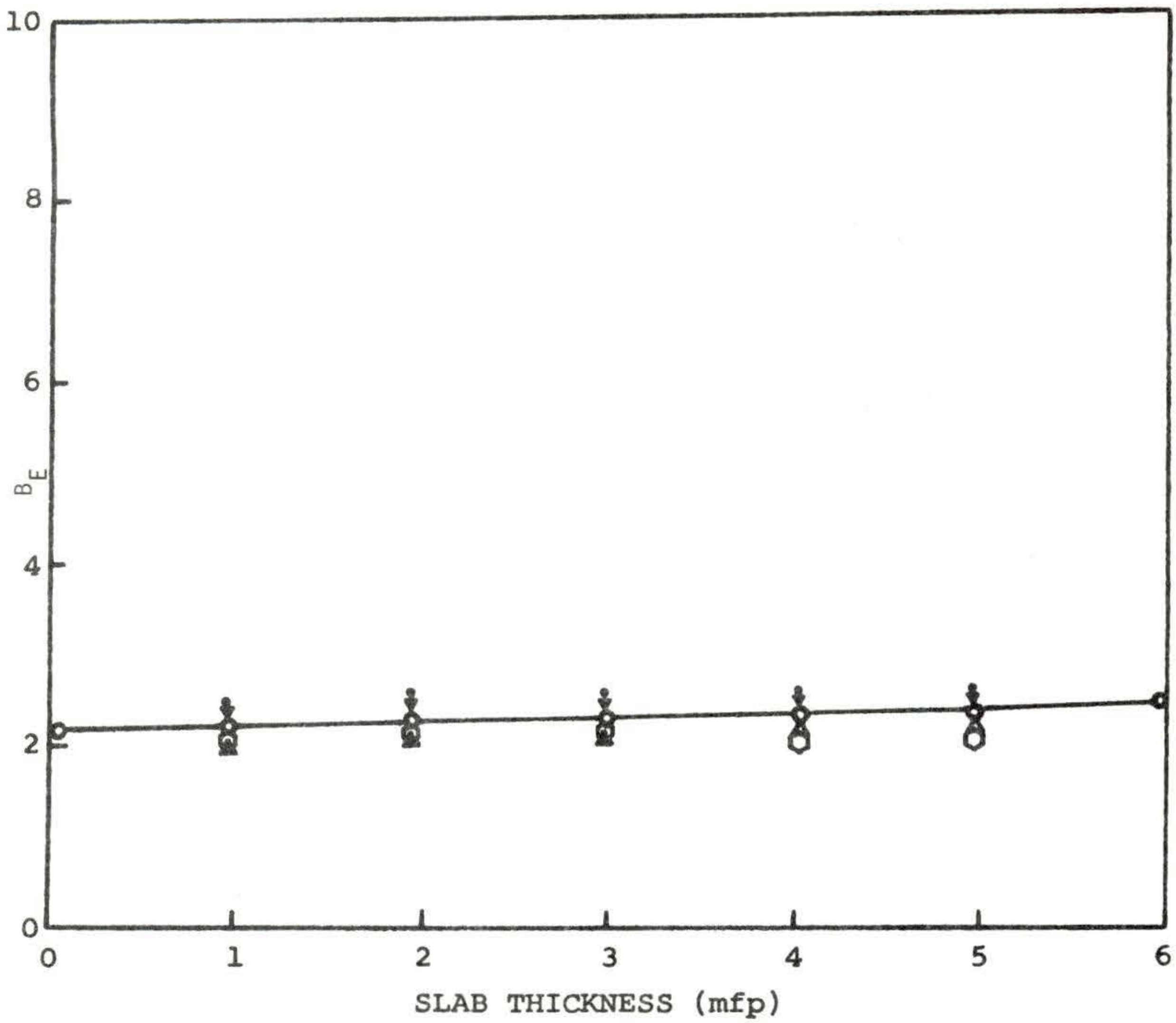
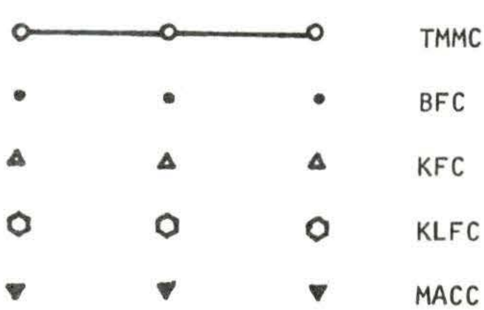



Figure 49. Energy fluence buildup factor of a 6 mfp lead-uranium shield for a 1 Mev point isotropic source



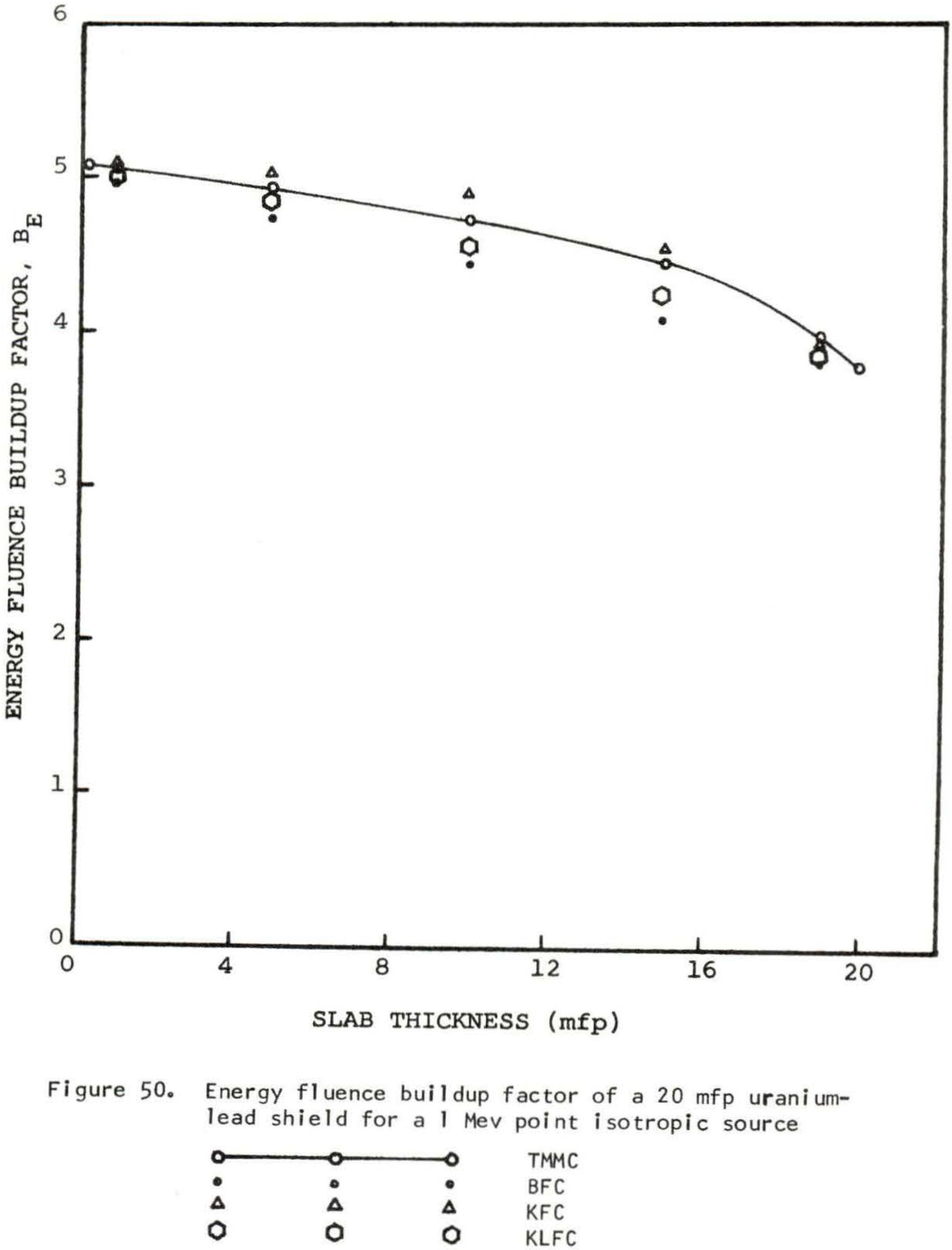



Figure 50. Energy fluence buildup factor of a 20 mfp uranium-lead shield for a 1 Mev point isotropic source

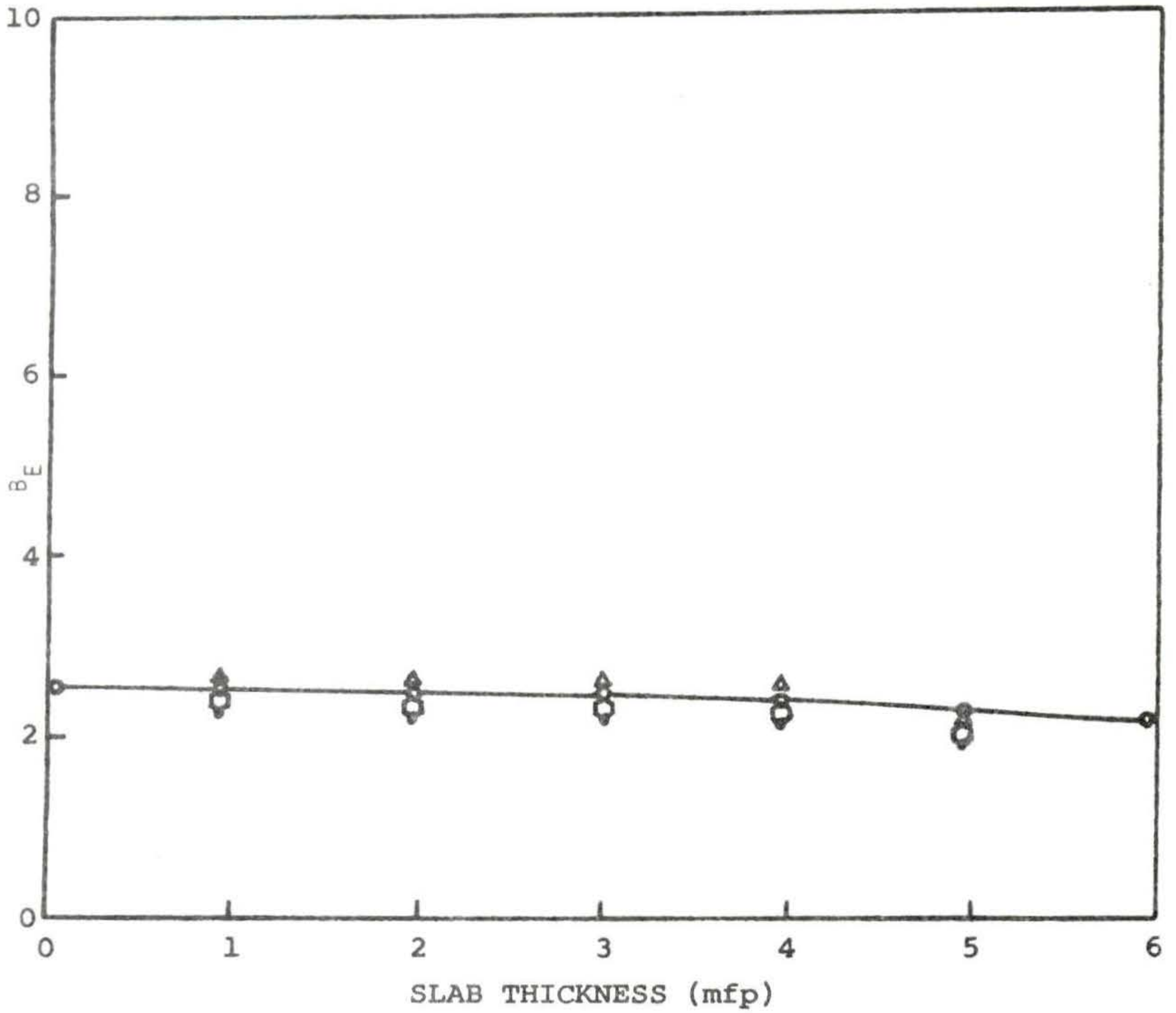
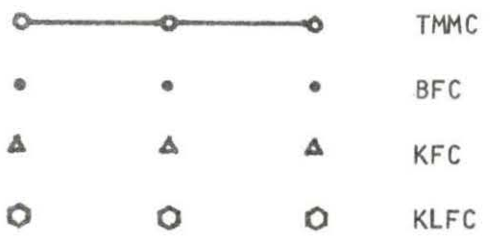



Figure 51. Energy fluence buildup factor of a 6 mfp uranium-lead shield for a 1 Mev point isotropic source



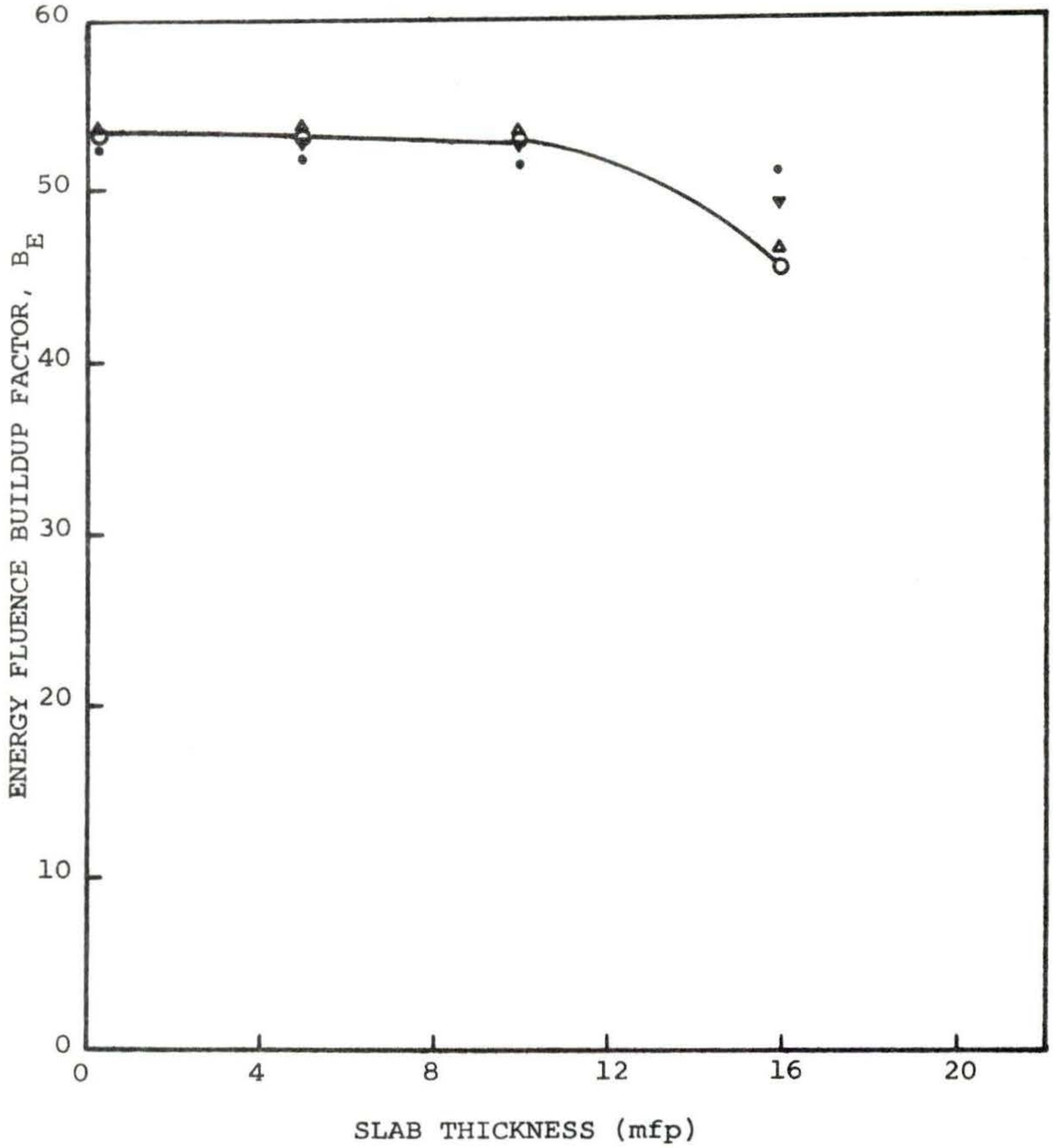
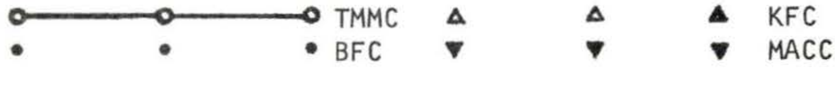



Figure 52. Energy fluence buildup factor, $B_E$, of a 20 mfp water-aluminum-water system for a 1 Mev point isotropic source. Aluminum layer thickness = 4 mfp



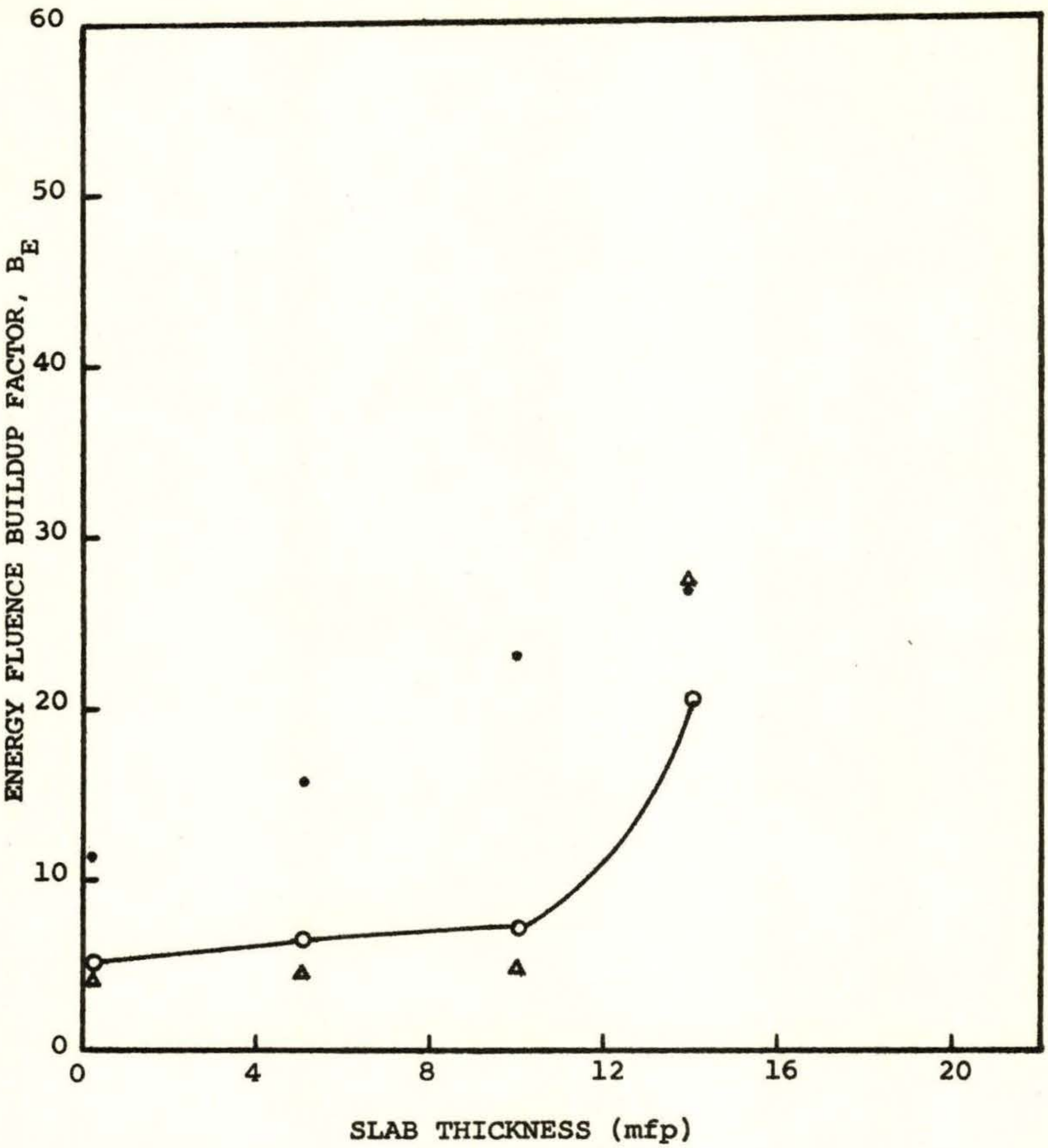



Figure 57. Energy fluence buildup factor, $B_E$, of a 20 mfp lead-water-lead system for a 1 Mev point isotropic source. Water layer thickness = 6 mfp

o———o———o TMMC
• • • BFC
Δ Δ Δ KFC

at each layer as the sum of individual difference in the buildup. Consequently, the buildup factors calculated by Broder's form are shown increasing almost linearly as the first (light) layer increases in thickness. In the actual cases, the buildup factors remain relatively constant until the last layer becomes relatively thin compared to the system thickness. Deviations of Broder's form due to this cause are observed in both the 20 mfp and the 6 mfp cases.

It is observed that the buildup factors calculated by the Kitazume form are generally in good agreement with those of the transmission matrix method. The formula of Kitazume approximates the "dominance by the last layer" effect by multiplying an exponential term, $\exp(-\alpha X_2)$, to the Broder's form [Equations (26) and (23)]. When the last (heavy) layer is relatively thick, the exponential term is zero for all practical purposes and hence the two-layer buildup factor of a light-heavy system is essentially the buildup factor of the last layer. As the last layer becomes relatively thin, the exponential function increases rapidly and the results resemble the transmission matrix method results. Mathematically speaking, for every transmission matrix method value of buildup factors, one can always fit an $\alpha$ to get the identical buildup factor by Kitazume form. However, very little information on $\alpha$ is available. The value of $\alpha$ in this investigation is chosen iteratively such that one single value of $\alpha$ would be sufficient and suitable for a two-layer system of fixed total thickness. In general, $\alpha = 1.0$ [15] would generate close agreement by the Kitazume form for a light-heavy system. Such close agreement between the

Kitazume form and the transmission matrix methods are observed in both the 20 mfp and 6 mfp cases.

Kalos form [see Equation (30)] was extended to apply to cases other than the water-lead combination and total thickness of more than 6 mfp. Generally speaking, the deviations of the results of Kalos' form were less in the 6 mfp cases than in the 20 mfp cases. For the particular cases of water followed by lead and water followed by uranium, the results from Kalos form showed very good agreement with those of the transmission matrix method. Application of Kalos form to other light-heavy combinations would result in a lower buildup value than the transmission matrix method. For such cases, the formula of Kitazume appeared to be superior.

The method of analytical continuation was applied to all light-heavy combinations except those involving lead and uranium as the second layer. The reason being that the first procedure in the method of analytical continuation is to convert the first layer into a hypothetical layer of the second medium giving an equivalent buildup factor. To achieve this, one would need to extrapolate the buildup value of lead and uranium well over 20 mfp where data would not have been calculated. In general, the results of the method of analytical continuation show poor agreement with the transmission matrix method results. The deviation of this method from the transmission matrix method is most notable when the differences between the two layers are most acute. This is so because the more drastic the change in energy spectrum and angular distribution that occurs at the interfacial boundary, the larger the introduced error. Deviations between the results of the method of analytical continuation and those of the transmission matrix method are observed in both the 20 mfp and the 6 mfp cases.

2. Heavy-light systems

For heavy-light combinations, the effect of "dominance by the last layer" is much less than the light-heavy combinations. In fact, for the cases where the first layer is lead or uranium, the buildup factor would undergo an almost linear decrease as the first layer increases in thickness. For such cases of linear decrease, the Broder form shows good agreement with the transmission matrix method for both the 20 mfp and the 6 mfp systems. For the cases where the first layer is iron or aluminum, the decrease of buildup is no longer linear and the Broder form is inadequate.

In the same manner as in the light-heavy systems, the formula of Kitazume describes the decrease of the buildup factors of the heavy-light systems by an exponential function, $\exp(-\alpha X_2)$. If the first layer is either lead or uranium, $\alpha$ is automatically set equal to zero since the buildup factor calculated under such condition is the closest one can get to the transmission matrix method value. This would reduce the Kitazume form into Broder form. When the first layer is either aluminum or iron, the Kitazume form approximates the transmission matrix method result just as in the previous cases. In general, the Kitazume form can describe the heavy-light systems adequately by choosing the best fit $\alpha$ value.

Except in the cases where lead or uranium is the first layer material, Kalos form [Equation (29)] is inadequate to describe the buildup behavior of a heavy-light combination. For the cases where the total thickness of the systems are 20 mfp, the deviations between the Kalos form and the transmission matrix method are especially apparent. The formula of Kalos is quite applicable to two-layer shields of total thickness less than 6 mfp and with either lead or uranium as the first layer.

For heavy-light systems, the method of analytical continuation generally shows good agreement with the transmission matrix method. It should be pointed out that there probably is error introduced when one reads a hypothetical thickness off the buildup factor graph. This error of reading off the graph may counter or compound the error due to the energy spectrum and angular distribution change at the interfacial boundary.

3. <u>The lead and uranium combinations</u>

Since both lead (Z=82) and uranium (Z=92) are heavy materials, their gamma energy spectra and angular distributions are similar in nature. The buildup factor undergoes relatively little change in such systems and all four formulas appear to describe the buildup behavior well. No distinction can be made in pinpointing which form is more accurate for the systems.

The various three-layer systems can be classified into two categories:

(1) Light material slab sandwiched in a heavy medium of fixed total thickness, e.g., the iron-water-iron system, etc.

(2) Heavy material slab sandwiched in a light medium of fixed total thickness, e.g., the water-lead-water system, etc.

The three-layer buildup factors evaluated by the transmission matrix method were compared with those of the formulas of Broder, Kitazume and analytical continuation. Since Kalos form was devised for a two-layer system, it was not used here for comparison.

The comparisons of three-layer buildup factors between the transmission matrix method and the three semi-empirical forms were shown in Figures 52 - 57. In general, Kitazume form showed the least deviation from the transmission matrix method. For systems of water slab moving in aluminum or iron (and reversed) the deviations of Broder form and the method of analytical continuation from the transmission matrix method were within reasonable limit. However, for the system of water-lead-water, Broder form was far from being accurate and the method of analytical continuation was inapplicable. The reason behind the inapplicability of the method of analytical continuation in this instance is the same as in the two-layer case where lead is the second layer (see page 77).

# V. CONCLUSIONS

It has been shown that the transmission matrix method can be used to generate buildup factors systematically as a function of source-detector distance, source wavelength and shielding material. Through comparisons with the Goldstein and Wilkins' results and considerations of the uncertainty in the original data, it is agreed that the transmission matrix method is both accurate and suitable for parametric studies of buildup factors.

Close agreement between the Goldstein and Wilkins' report and the transmission matrix method calculations for single layer materials of infinite medium was observed. Since the technique of transmission matrix method used to calculate the infinite buildup factors applied to the finite buildup factors in the same way, it is concluded that the finite single layer buildup factors evaluated by the transmission matrix method are also accurate within the uncertainty limit.

Due to the fact that there is no established benchmark value for multilayer buildup factors, no absolute statement can be made regarding the relative accuracies of the formulas of Broder, Kitazume, Kalos and the method of analytical continuation. It is observed, however, that the results of all four formulas deviate somewhat from the transmission matrix method calculations. For light-heavy systems, Kitazume form offers the least deviations from the transmission matrix method. For heavy-light systems, least deviations is best achieved by using a combination of Broder and Kitazume form. For three-layer systems, Kitazume form shows the least deviations from the transmission matrix method calculations among the three applicable formulas.

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# VII. APPENDIX A

## Definitions:

Roentgen: A roentgen is defined as the quantity of X or gamma radiation such that the associated corpuscular emission per 0.001293 gm of air produces ions carrying 1 e.s.u. of electricity of either sign. This is equivalent to $2.58 \times 10^{-4}$ coulomb/kg.

$1\ R = 1\ e.s.u./0.001293\ gm\ of\ air = 2.58 \times 10^{-4}\ coulomb/kg.$

Rad: A rad is defined as the quantity of ionization radiation that imparted 100 erg of energy to the matter in a volume element of mass 1 gm.

$1\ rad = 100\ erg/g = 10^{-2}\ joule/kg$

## VIII. APPENDIX B

### Transmission Matrix Theory [21] :

Consider n laminated layers of the same material of infinitesimal thickness $t_1$, $t_2$, ... $t_n$, one obtains

$$H(t) = H(t_1 + t_2 + \ldots t_n) = H(t_n)\, H(t_{n-1}) \ldots H(t_1) = \prod_{i=n}^{1} H(t_i) \ldots \quad \text{(B1)}$$

$$\therefore H(t) = e^{-wt} = 1 - wt + \frac{(wt)^2}{2!} + \ldots \quad \text{(B2)}$$

where

$$t = \sum_{i=1}^{n} t_i \quad .$$

The transmission matrix operator, $T(t)$ and reflection operator, $R(t)$ can be expanded as follows:

$$T(t) = e^{-\alpha t} = 1 - \alpha t + \frac{(\alpha t)^2}{2!} + \ldots \quad \text{(B3)}$$

$$R(t) = 1 - e^{-\beta t} = \beta t - \frac{(\beta t)^2}{2!} + \ldots \quad . \quad \text{(B4)}$$

Recall from Equation (10), one has the relation:

$$H(t) = \begin{bmatrix} T - RT^{-1}R & RT^{-1} \\ -T^{-1}R & T^{-1} \end{bmatrix} \quad \text{(B5)}$$

Substituting Equations (B3) and (B4) into Equation (B5), one obtains

$$H(t) = I - \begin{bmatrix} \alpha & -\beta \\ \beta & -\alpha \end{bmatrix} t + \cdots\cdots \tag{B6}$$

Equating Equations (B2) and (B6), one finds

$$W = \begin{bmatrix} \alpha & -\beta \\ \beta & -\alpha \end{bmatrix} . \tag{B7}$$

For the purpose of computational convenience it is necessary to diagonalize the matrix W. To achieve diagonalization, first matrix W is transformed to $\overline{W}$ by:

$$\overline{W} = P^{-1} WP = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & \zeta \\ \delta & 0 \end{bmatrix} \tag{B8}$$

where

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = 2P^{-1}$$

$$\zeta = \alpha + \beta$$

$$\delta = \alpha - \beta \quad .$$

The second step is the solution of the eigenvalue problem of $\overline{W}$. Consider:

$$\overline{W}Z = Z\Lambda \tag{B9}$$

where $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ $\quad \Lambda = \begin{bmatrix} \Lambda_{11} & \theta \\ \theta & \Lambda_{22} \end{bmatrix}$ .

From Equation (B9), one obtains the following four equations:

$$\zeta Z_{21} = Z_{11}\ \Lambda_{11} \tag{B10}$$

$$\zeta Z_{22} = Z_{12}\ \Lambda_{22} \tag{B11}$$

$$\xi Z_{11} = Z_{21}\ \Lambda_{11} \tag{B12}$$

$$\xi Z_{12} = Z_{22}\ \Lambda_{22} \tag{B13}$$

It is observed from Equations (B10) and (B13) that if X is column eigenvector of the matrix A, where $A = \zeta\,\xi$, then it follows:

$$Z_{11} = Z_{12} = X \tag{B14}$$

$$Z_{21} = \xi\ X\ \Gamma^{-1} \tag{B15}$$

$$Z_{22} = -\xi\ X\ \Gamma^{-1} \tag{B16}$$

where $\Lambda_{11} = \Gamma$, $\Lambda_{22} = -\Gamma$. Substituting Equation (B14) - (B16) into Equation (B9), one obtains:

$$\begin{bmatrix} \Theta & \zeta \\ \zeta & \Theta \end{bmatrix} \begin{bmatrix} X & X \\ \xi X\Gamma^{-1} & -\xi X\Gamma^{-1} \end{bmatrix} = \begin{bmatrix} X & X \\ \xi X\Gamma^{-1} & -\xi X\Gamma^{-1} \end{bmatrix} \begin{bmatrix} \Gamma & \Theta \\ \Theta & -\Gamma \end{bmatrix} . \tag{B17}$$

Taking the transpose of Equation (B9), one obtains a similar relation for Y, the row eigenvector of matrix A.

$$\overline{W}^T\ U^T = U^T\ \Lambda^T \tag{B18}$$

$$\therefore \quad U\ \overline{W} = \Lambda\ U \tag{B19}$$

where

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \Lambda_{11} & \Theta \\ \Theta & \Lambda_{22} \end{bmatrix} .$$

From Equation (B19), one obtains the following four equations:

$$U_{12}\ \delta = \Lambda_{11}\ U_{11} \qquad \text{(B20)}$$

$$U_{22}\ \delta = \Lambda_{22}\ U_{21} \qquad \text{(B21)}$$

$$U_{11}\ \zeta = \Lambda_{11}\ U_{12} \qquad \text{(B22)}$$

$$U_{21}\ \zeta = \Lambda_{22}\ U_{22} \qquad \text{(B23)}$$

From Equation (B20) - (B23), it can be shown that the following is true:

$$U_{11} = U_{21} = Y \qquad \text{(B24)}$$

$$U_{22} = \Gamma^{-1}\ Y\ \zeta \qquad \text{(B25)}$$

$$U_{12} = \Gamma^{-1}\ Y\ \zeta \qquad \text{(B26)}$$

Having had the column eigenvector X, and row eigenvector, Y, defined, and choosing YX = 1, the diagonalization of matrix W proceeds as follows:

$$U\bar{W}Z = \begin{bmatrix} Y & \Gamma^{-1}Y\zeta \\ Y & -\Gamma^{-1}Y\zeta \end{bmatrix} \begin{bmatrix} \theta & \zeta \\ \delta & \theta \end{bmatrix} \begin{bmatrix} X & X \\ \delta X\Gamma^{-1} & -\delta X\Gamma^{-1} \end{bmatrix} . \qquad \text{(B27)}$$

Substituting Equation (B17) into (B27), the above equation becomes

$$U\bar{W}Z = \begin{bmatrix} Y & \Gamma^{-1}Y\zeta \\ Y & -\Gamma^{-1}Y\zeta \end{bmatrix} \begin{bmatrix} X & X \\ \delta X\Gamma^{-1} & -\delta X\Gamma^{-1} \end{bmatrix} \begin{bmatrix} \Gamma & \theta \\ \theta & -\Gamma \end{bmatrix}$$

$$= 2 \begin{bmatrix} I & \theta \\ \theta & I \end{bmatrix} \begin{bmatrix} \Gamma & \theta \\ \theta & -\Gamma \end{bmatrix} . \qquad \text{(B28)}$$

Substituting Equation (B8) into (B27), one obtains the following:

$$U\bar{W}Z = \frac{1}{2}\begin{bmatrix} Y & \Gamma^{-1}Y\zeta \\ Y & -\Gamma^{-1}Y\zeta \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} \alpha & -\beta \\ \beta & -\alpha \end{bmatrix}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}\begin{bmatrix} X & X \\ \delta X\Gamma^{-1} & -\delta X\Gamma^{-1} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} Y + \Gamma^{-1}Y\zeta & Y - \Gamma^{-1}Y\zeta \\ Y - \Gamma^{-1}Y\zeta & Y + \Gamma^{-1}Y\zeta \end{bmatrix}\begin{bmatrix} \alpha & -\beta \\ \beta & -\alpha \end{bmatrix}\begin{bmatrix} X + \delta X\Gamma^{-1} & X - \delta X\Gamma^{-1} \\ X - \delta X\Gamma^{-1} & X + \delta X\Gamma^{-1} \end{bmatrix} \quad \text{(B29)}$$

Equating Equations (B28) and (B29), the diagonalization relation of matrix W is:

$$\frac{1}{4}\begin{bmatrix} Y + \Gamma^{-1}Y\zeta & Y - \Gamma^{-1}Y\zeta \\ Y - \Gamma^{-1}Y\zeta & Y + \Gamma^{-1}Y\zeta \end{bmatrix}\begin{bmatrix} \alpha & -\beta \\ \beta & -\alpha \end{bmatrix}\begin{bmatrix} X + \delta X\Gamma^{-1} & X-\delta X\Gamma^{-1} \\ X - \delta X\Gamma^{-1} & X+\delta X\Gamma^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma & \theta \\ \theta & -\Gamma \end{bmatrix} \quad \text{(B30)}$$

Let

$$C \pm = Y \pm \Gamma^{-1}\, Y\zeta \qquad B \pm = X \pm \delta\, X\, \Gamma^{-1}$$

Equation (B30) becomes:

$$\frac{1}{4}\begin{bmatrix} C+ & C- \\ C- & C+ \end{bmatrix} W\begin{bmatrix} B+ & B- \\ B- & B+ \end{bmatrix} = \begin{bmatrix} \Gamma & \theta \\ \theta & -\Gamma \end{bmatrix} \quad \text{(B31)}$$

Having diagonalized the matrix W, the next step is to relate Equation (B31) to the matrix H(t) and find the transmission and reflection matrix operators, T(t) and R(t), respectively. From Equation (B2), one derives the following relations:

$H(t) = e^{-Wt}$, if W can be diagonalized to $SWS^{-1}$, then:

$$SH(t)\, S^{-1} = S\, (e^{-Wt})\, S^{-1} = e^{-SWS^{-1}\, t}$$

$$\therefore \quad H(t) = S^{-1}\, e^{-SWS^{-1} t}\, S \tag{B32}$$

Note that Equation (B31) can be substituted into Equation (B32) by setting:

$$S = \frac{1}{2} \begin{bmatrix} C+ & C- \\ C- & C+ \end{bmatrix} \qquad S^{-1} = \frac{1}{2} \begin{bmatrix} B+ & B- \\ B- & B+ \end{bmatrix} \tag{B33}$$

$$\therefore \quad e^{-SWS^{-1} t} = \exp - \frac{1}{4} \begin{bmatrix} C+ & C- \\ C- & C+ \end{bmatrix} W \begin{bmatrix} B+ & B- \\ B- & B+ \end{bmatrix} t$$

$$= \exp - \begin{bmatrix} \Gamma & \theta \\ \theta & -\Gamma \end{bmatrix} t = \begin{bmatrix} e^{-\Lambda t} & \theta \\ \theta & e^{\Lambda t} \end{bmatrix} \tag{B34}$$

Substituting Equations (B33) and (B34) into Equation (B32), one obtains the relation:

$$H(t) = \frac{1}{4} \begin{pmatrix} B+ & B- \\ B- & B+ \end{pmatrix} \begin{pmatrix} e^{-\Lambda t} & \theta \\ \theta & e^{\Lambda t} \end{pmatrix} \begin{pmatrix} C+ & C- \\ C- & C+ \end{pmatrix} .$$

$$H(t) = \frac{1}{4} \begin{bmatrix} B_+ e^{-\Lambda t} C_+ + B_- e^{-\Lambda t} C_- & B_+ e^{-\Lambda t} C_- + B_- e^{\Lambda t} C_+ \\ B_- e^{-\Lambda t} C_+ + B_+ e^{\Lambda t} C_- & B_- e^{-\Lambda t} C_- + B_+ e^{\Lambda t} C_+ \end{bmatrix} \tag{B35}$$

Equating Equations (B35) and (B5), the expression of T(t) and R(t) are found as follows:

$$T(t) = 4(B_- e^{-\Lambda t} C_- + B_+ e^{\Lambda t} C_+)^{-1} \quad \text{(B36)}$$

$$R(t) = \frac{1}{4} (B_+ e^{-\Lambda t} C_- + B_- e^{\Lambda t} C_+) T(t) \quad \text{(B37)}$$

With T(t) and R(t) calculated, the total transmitted gamma flux can be computed according to Equation (9). The following procedures summarize the algorithm of transmission matrix mathod in calculating buildup factors:

(1) Input of data in photoelectric and pair production cross section, atomic number and density, to form the matrices $\alpha$ and $\beta$. Matrices $\zeta$, $\delta$ and A were subsequently formed.

(2) Solution of the eignvalue problems to find the column and row eignvectors, X and Y, and the eignvalue $\Gamma$ of the matrix A.

(3) Compute the values of $C_{\pm}$ and $B_{\pm}$ from X, Y, $\Gamma$ obtained in step (2).

(4) Compute the transmission and reflection matrix operator, T(t) and R(t), and finally the total transmitted gamma flux spectrum and the buildup factors.

It should be noted that steps (1) through (3) are independent on the geometric configuration and thickness of the medium. They are only related to the cross sections of the medium. Thus the first three steps need to be done only once for each material and could be stored for future usage. This unique feature of transmission matrix method algorithm makes it particularly suited for parametric studies in buildup factors.

# IX. ACKNOWLEDGMENTS

The author wishes to express his gratitude to Dr. A. F. Rohach for his guidance and encouragement offered during the course of this investigation.

This work was supported by the Engineering Research Institute, Iowa State University, through funds made available by the Atomic Energy Commission.